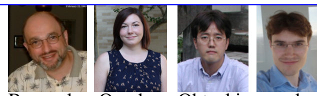
Cars, Interchanges, Traffic Counters, and some Pretty Darned Good Knot Invariants

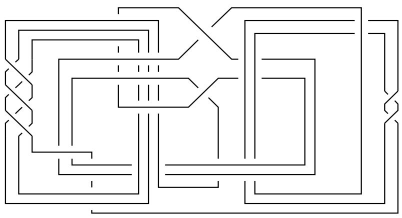
More at ωεβ/APAI

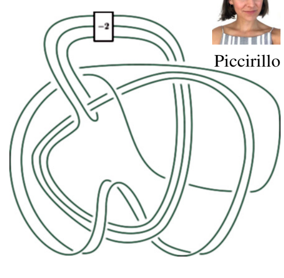
Abstract. Reporting on joint work with Roland van der Veen, I'll tell you some stories about ρ_1, an easy to define, strong, fast to compute, homomorphic, and well-connected knot invariant. ρ_1 was first studied by Rozansky and Overbay [Ro1, Ro2, Ro3, Ov] and Ohtsuki [Oh2], it has far-reaching generalizations, it is elementary and dominated by the coloured Jones polynomial, and I wish I understood it.
Common misconception. Dominated, elementary ⇒ lesser.


Rozansky Overbay Ohtsuki van der Veen

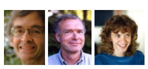
We seek strong, fast, homomorphic knot and tangle invariants.
Strong. Having a small "kernel".
Fast. Computable even for large knots (best: poly time).


Piccirillo

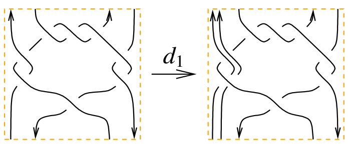




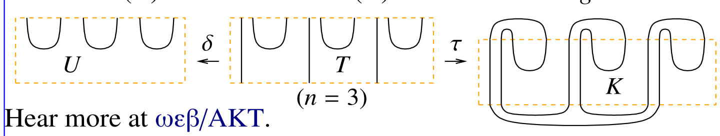
Gompf–Scharlemann–Thompson



Homomorphic. Extends to tangles and behaves under tangle operations; especially gluings and doublings:



Why care for "Homomorphic"? Theorem. A knot K is *ribbon* iff there exists a $2n$-component tangle T with skeleton as below such that $\tau(T) = K$ and where $\delta(T) = U$ is the *untangle*:



Hear more at ωεβ/AKT.

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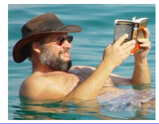
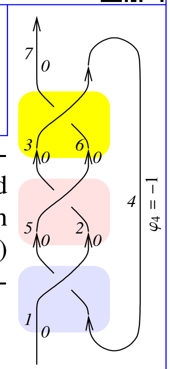
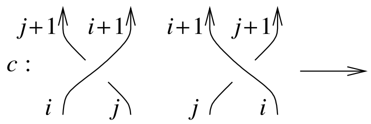


Jones: Formulas stay; interpretations change with time.



Formulas. Draw an n-crossing knot K as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \dots, 2n+1\}$ and with rotation numbers φ_k. Let A be the $(2n+1)\times(2n+1)$ matrix constructed by starting with the identity matrix I, and adding a 2×2 block for each crossing:



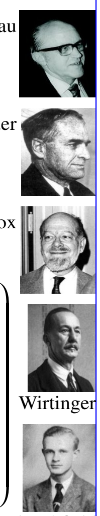
A	col $i+1$	col $j+1$
row i	$-T^s$	$T^s - 1$
row j	0	-1

Let $G = (g_{\alpha\beta}) = A^{-1}$. For the trefoil example, it is:

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$


"The Green Function"


Burau Alexander Fox Wirtinger Blanchfield

Note. The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2}\det(A), \qquad \text{with } \varphi = \sum_k \varphi_k,\ w = \sum_c s.$$

Classical Topologists: This is boring. Yawn.

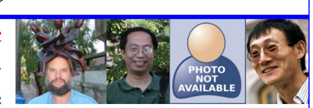
Formulas, continued. Finally, set

$$R_1(c) := s\left(g_{ji}\left(g_{j+1,j} + g_{j,j+1} - g_{ij}\right) - g_{ii}\left(g_{j,j+1} - 1\right) - 1/2\right)$$

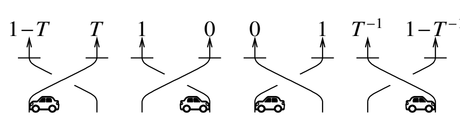
$$\rho_1 := \Delta^2\left(\sum_c R_1(c) - \sum_k \varphi_k\,(g_{kk} - 1/2)\right).$$

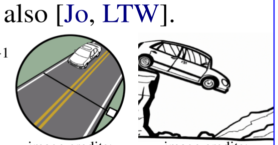
In our example $\rho_1 = -T^2 + 2T - 2 + 2T^{-1} - T^{-2}$.
Theorem. ρ_1 is a knot invariant. Proof: later.
Classical Topologists: Whiskey Tango Foxtrot?

Cars, Interchanges, and Traffic Counters. Cars always drive forward. When a car crosses over a bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0^*$. At the very end, cars fall off and disappear. See also [Jo, LTW].


Jones Lin Tian Wang


$p = 1 - T^s$

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* In algebra $x \sim 0$ if for every y in the ideal generated by x, $1 - y$ is invertible.

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Video: http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210. Handout: http://www.math.toronto.edu/~drorbn/Talks/Nara-2308.