A Small-Print Page on ρ_d, $d>1$.

Definition. $\langle f(z_i), h(\zeta_i)\rangle_{\{z_i\}} := f(\partial_{\zeta_i})h\big|_{\zeta_i=0}$, so $\langle p^2x^2, e^{g\pi\xi}\rangle = 2g^2$.

Baby Theorem. There exist (non unique) power series $r^{\pm}(p_1,p_2,x_1,x_2) = \sum_d \epsilon^d r_d^{\pm}(p_1,p_2,x_1,x_2) \in \mathbb{Q}[T^{\pm1},p_1,p_2,x_1,x_2][\![\epsilon]\!]$ with $\deg r_d^{\pm} \le 2d+2$ ("docile") such that the power series $Z^b = \sum \rho_d^b \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i,p_j,x_i,x_j)\right), \exp\left(\sum_{\alpha,\beta} g_{\alpha\beta}\pi_\alpha\xi_\beta\right)\right\rangle_{\{p_\alpha,x_\beta\}}$$

is a bnot invariant. Beyond the once-and-for-all computation of $g_{\alpha\beta}$ (a matrix inversion), Z^b is computable in $O(n^d)$ operations in the ring $\mathbb{Q}[T^{\pm1}]$.

(Bnots are knot diagrams modulo the braid-like Reidemeister moves, but not the cyclic ones).

Theorem. There also exist docile power series $\gamma^{\varphi}(\bar p,\bar x) = \sum_d \epsilon^d \gamma_d^{\varphi} \in \mathbb{Q}[T^{\pm1},\bar p,\bar x][\![\epsilon]\!]$ such that the power series $Z = \sum \rho_d \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i,p_j,x_i,x_j) + \sum_k \gamma^{\varphi_k}(\bar p_k,\bar x_k)\right),\right.$$
$$\left.\exp\left(\sum_{\alpha,\beta} g_{\alpha\beta}(\pi_\alpha+\bar\pi_\alpha)(\xi_\beta+\bar\xi_\beta) + \sum_\alpha \pi_\alpha\bar\xi_\alpha\right)\right\rangle_{\{p_\alpha,\bar p_\alpha,x_\beta,\bar x_\beta\}}$$

is a knot invariant, as easily computable as Z^b.

Implementation. Data, then program (with output using the Conway variable $z=\sqrt{T}-1/\sqrt{T}$), and then a demo. See `Rho.nb` of ωεβ/ap.

```
V@γ1,φ_[k_] = φ (1/2 - p̄k x̄k); V@γ2,φ_[k_] = -φ^2 p̄k x̄k / 2;
V@γ3,φ_[k_] := -φ^3 p̄k x̄k / 6
```

```
V@r1,s_[i_, j_] :=
 s (-1 + 2 pi xi - 2 pj xi + (-1 + T^s) pi pj xi^2 + (1 - T^s) pj^2 xi^2 - 2 pi pj xi xj + 2 pj^2 xi xj) / 2
```

```
V@r2,1[i_, j_] :=
 (-6 pi xi + 6 pj xi - 3 (-1 + 3 T) pi pj xi^2 + 3 (-1 + 3 T) pj^2 xi^2 + 4 (-1 + T) pi^2 pj xi^3 -
   2 (-1 + T) (5 + T) pi pj^2 xi^3 + 2 (-1 + T) (3 + T) pj^3 xi^3 + 18 pi pj xi xj -
   18 pj^2 xi xj - 6 pi^2 pj xi^2 xj + 6 (2 + T) pi pj^2 xi^2 xj - 6 (1 + T) pj^3 xi^2 xj -
   6 pi pj^2 xi xj^2 + 6 pj^3 xi xj^2) / 12
```

```
V@r2,-1[i_, j_] :=
 (-6 T^2 pi xi + 6 T^2 pj xi + 3 (-3 + T) T pi pj xi^2 - 3 (-3 + T) T pj^2 xi^2 -
   4 (-1 + T) T pi^2 pj xi^3 + 2 (-1 + T) (1 + 5 T) pi pj^2 xi^3 - 2 (-1 + T) (1 + 3 T) pj^3 xi^3 +
   18 T^2 pi pj xi xj - 18 T^2 pj^2 xi xj - 6 T^2 pi^2 pj xi^2 xj + 6 T (1 + 2 T) pi pj^2 xi^2 xj -
   6 T (1 + T) pj^3 xi^2 xj - 6 T^2 pi pj^2 xi xj^2 + 6 T^2 pj^3 xi xj^2) / (12 T^2)
```

```
V@r3,1[i_, j_] :=
 (4 pi xi - 4 pj xi + 2 (5 + 7 T) pi pj xi^2 - 2 (5 + 7 T) pj^2 xi^2 - 4 (-5 + 6 T) pi^2 pj xi^3 +
   4 (-16 + 17 T + 2 T^2) pi pj^2 xi^3 - 4 (-11 + 11 T + 2 T^2) pj^3 xi^3 + 3 (-1 + T) pi^3 pj xi^4 -
   3 (-1 + T) (4 + 3 T) pi^2 pj^2 xi^4 + (-1 + T) (13 + 22 T + T^2) pi pj^3 xi^4 -
   (-1 + T) (4 + 13 T + T^2) pj^4 xi^4 - 28 pi pj xi xj + 28 pj^2 xi xj + 36 pi^2 pj xi^2 xj -
   12 (9 + 2 T) pi pj^2 xi^2 xj + 24 (3 + T) pj^3 xi^2 xj - 4 pi^3 pj xi^3 xj + 28 T pi^2 pj^2 xi^3 xj -
   4 (-6 + 17 T + T^2) pi pj^3 xi^3 xj + 4 (-5 + 10 T + T^2) pj^4 xi^3 xj + 24 pi pj^2 xi xj^2 -
   24 pj^3 xi xj^2 - 24 pi^2 pj^2 xi^2 xj^2 + 6 (10 + T) pi pj^3 xi^2 xj^2 - 6 (6 + T) pj^4 xi^2 xj^2 -
   4 pi pj^3 xi xj^3 + 4 pj^4 xi xj^3) / 24
```

```
V@r3,-1[i_, j_] :=
 (-4 T^3 pi xi + 4 T^3 pj xi - 2 T^2 (7 + 5 T) pi pj xi^2 + 2 T^2 (7 + 5 T) pj^2 xi^2 -
   4 T^2 (-6 + 5 T) pi^2 pj xi^3 + 4 T (-2 - 17 T + 16 T^2) pi pj^2 xi^3 -
   4 T (-2 - 11 T + 11 T^2) pj^3 xi^3 + 3 (-1 + T) T^2 pi^3 pj xi^4 - 3 (-1 + T) T (3 + 4 T) pi^2 pj^2 xi^4 +
   (-1 + T) (1 + 22 T + 13 T^2) pi pj^3 xi^4 - (-1 + T) (1 + 13 T + 4 T^2) pj^4 xi^4 +
   28 T^3 pi pj xi xj - 28 T^3 pj^2 xi xj - 36 T^3 pi^2 pj xi^2 xj + 12 T^2 (2 + 9 T) pi pj^2 xi^2 xj -
   24 T^2 (1 + 3 T) pj^3 xi^2 xj + 4 T^3 pi^3 pj xi^3 xj - 28 T^2 pi^2 pj^2 xi^3 xj -
   4 T (-1 - 17 T + 6 T^2) pi pj^3 xi^3 xj + 4 T (-1 - 10 T + 5 T^2) pj^4 xi^3 xj -
   24 T^3 pi pj^2 xi xj^2 + 24 T^3 pj^3 xi xj^2 + 24 T^3 pi^2 pj^2 xi^2 xj^2 - 6 T^2 (1 + 10 T) pi pj^3 xi^2 xj^2 +
   6 T^2 (1 + 6 T) pj^4 xi^2 xj^2 + 4 T^3 pi pj^3 xi xj^3 - 4 T^3 pj^4 xi xj^3) / (24 T^3)
```

```
{p*, x*, p̄*, x̄*} = {π, ξ, π̄, ξ̄};  (z_i__)* := (z*)i;
Zip{}[ℰ_] := ℰ;
Zip{z_,zs___}[ℰ_] :=
 (Collect[ℰ // Zip{zs}, z] /. f_. z^d_. :> (D[f, {z*, d}])) /. z* → 0
```

```
gPair[fs_, w_] :=
 gPair[fs, w] =
  Collect[ZipJoin@@Table[{pα,p̄α,xα,x̄α},{α,w}][
    (Times @@ (V /@ fs))
     Exp[Sum[gα,β (πα + π̄α) (ξβ + ξ̄β), {α, w}, {β, w}] - Sum[ξ̄α πα, {α, w}]]],
   g__, Factor]
```

```
T2z[p_] := Module[{q = Expand[p], n, c},
  If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
   c z^(2 n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2 n)]]];
```

```
Z_d_[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, d1, Z1, Z2, Z3},
  {Cs, φ} = Rot[K]; n = Length[Cs]; A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1 ; 0  -1 ))];
  {Δ, G} = Factor@{T^((-Total[φ] - Total[Cs[[All,1]]])/2) Det@A, Inverse@A};
  Z1 =
   Exp[Total[Cases[Cs, {s_, i_, j_} :> Sum[ε^d1 rd1,s[i, j], {d1, d}]]] +
     Sum[ε^d1 γd1,φ[[k]][k], {k, 2 n}, {d1, d}] /. γ_,0[_] → 0];
  Z2 = Expand[F[{}, {}] × Normal@Series[Z1, {ε, 0, d}]] //.
    F[fs_, {es___}] × (f : (r | γ)ps__[is__])^p_. :>
     F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}];
  Z3 = Expand[Z2 /. F[fs_, es_] :> Expand[gPair[
        Replace[fs, Thread[es → Range@Length@es], {2}], Length@es
       ] /. gα_,β_ :> G[[es[[α]], es[[β]]]]]];
  Collect[{Δ, Z3 /. ε^p_. → p! Δ^(2 p) ε^p}, ε, T2z] ];
```

```
Z2[GST48] (* takes a few minutes *)
```

$\{1 - 4 z^2 - 61 z^4 - 207 z^6 - 296 z^8 - 210 z^{10} - 77 z^{12} - 14 z^{14} - z^{16},$
$1 + (38 z^2 + 255 z^4 + 1696 z^6 + 16281 z^8 + 86952 z^{10} + 259994 z^{12} + 487372 z^{14} + 615066 z^{16} + 543148 z^{18} + 341714 z^{20} + 153722 z^{22} + 48983 z^{24} + 10776 z^{26} + 1554 z^{28} + 132 z^{30} + 5 z^{32})\,\epsilon +$
$(-8 - 484 z^2 + 9709 z^4 + 165952 z^6 + 1590491 z^8 + 16256508 z^{10} + 115341797 z^{12} + 432685748 z^{14} + 395838354 z^{16} - 4017557792 z^{18} - 23300064167 z^{20} - 70082264972 z^{22} - 142572271191 z^{24} - 209475503700 z^{26} - 221616295209 z^{28} - 151502648428 z^{30} - 23700199243 z^{32} + 99462146328 z^{34} + 164920463074 z^{36} + 162550825432 z^{38} + 119164552296 z^{40} + 69153062608 z^{42} + 32547596611 z^{44} + 12541195448 z^{46} + 3961384155 z^{48} + 1021219696 z^{50} + 212773106 z^{52} + 35264208 z^{54} + 4537548 z^{56} + 436600 z^{58} + 29536 z^{60} + 1252 z^{62} + 25 z^{64})\,\epsilon^2\}$

```
TableForm[Table[Join[{K[[1]]K[[2]]}, Z3[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center] (* takes a few minutes *)
```

3_1	$1+z^2$	$1 + (2z^2+z^4)\epsilon + (2-4z^2+3z^4+4z^6+z^8)\epsilon^2 + (-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12})\epsilon^3$
4_1	$1-z^2$	$1 + (-2+2z^4)\epsilon^2$
5_1	$1+3z^2+z^4$	$1 + (10z^2+21z^4+12z^6+2z^8)\epsilon + (6-28z^2+33z^4+364z^6+655z^8+536z^{10}+227z^{12}+48z^{14}+4z^{16})\epsilon^2 + (-60+970z^2+645z^4-3380z^6-3280z^8+7470z^{10}+19475z^{12}+20536z^{14}+12564z^{16}+4774z^{18}+1109z^{20}+144z^{22}+8z^{24})\epsilon^3$
5_2	$1+2z^2$	$1 + (6z^2+5z^4)\epsilon + (4-20z^2+43z^4+64z^6+26z^8)\epsilon^2 + (-36+498z^2-883z^4+100z^6+816z^8+556z^{10}+146z^{12})\epsilon^3$
6_1	$1-2z^2$	$1 + (-2z^2+z^4)\epsilon + (-4+4z^2+25z^4-8z^6+2z^8)\epsilon^2 + (12+154z^2-223z^4-608z^6+100z^8-52z^{10}+10z^{12})\epsilon^3$
6_2	$1-z^2-z^4$	$1 + (-2z^2-3z^4+2z^6+z^8)\epsilon + (-2-4z^2+29z^4+28z^6+42z^8-8z^{10}-2z^{12}+4z^{14}+z^{16})\epsilon^2 + (12+166z^2+155z^4-194z^6-2453z^8-1622z^{10}-1967z^{12}-258z^{14}+49z^{16}-30z^{18}+z^{20}+6z^{22}+z^{24})\epsilon^3$
6_3	$1+z^2+z^4$	$1 + (2+8z^2-16z^6-24z^8-16z^{10}-2z^{12})\epsilon^2$

Video: http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210. Handout: http://www.math.toronto.edu/~drorbn/Talks/Nara-2308.