Computing the Zombian of an Unfinished Columbarium

Confession. It's about 50% of what I do.

Apology. It's a 20 minutes talk. Necessarily, it will be superficial.

Abstract. The zombies need to compute a quantity, the zombian, that pertains to some structure — say, a columbarium. But unfortunately (for them), a part of that structure will only be known in the future. What can they compute today with the parts they already have to hasten tomorrow's computation?

That's a common quest, and I will illustrate it with a few examples from knot theory and with two examples about matrices — determinants and signatures. I will also mention two of my dreams (perhaps delusions): that one day I will be able to reproduce, and extend, the Rolfsen table of knots using code of the highest level of beauty.

Columbaria in an East Sydney Cemetery

Jacobian, Hamiltonian, Zombian

Image: Freepik.com

Computing Zombians of Unfinished Columbaria.

- Future zombies must be able to complete the computation.
- Must be no slower than for finished ones.
- Future zombies must not even know the size of the task that today's zombies were facing.
- We must be able to extend to ZPUCs, Zombie Processed Unfinished Columbaria!

Zombies: Freepik.com

Columbarium near Assen

Exercise 1. Compute the sum of 1,000 numbers, the last 50 of which are still unknown.

Exercise 2. Compute the determinant of a $1,000 \times 1,000$ matrix in which 50 entries are not yet given.

Example 3. Same, for signatures of matrices / quadratic forms.

A quadratic form on a v.s. V over $\mathbb{C}$ is a quadratic $Q\colon V \to \mathbb{C}$, or a sesquilinear Hermitian $\langle\cdot,\cdot\rangle$ on $V \times V$ (so $\langle x,y\rangle = \overline{\langle y,x\rangle}$ and $Q(y) = \langle y,y\rangle$), or given a basis η_i of V^*, a matrix $A = (a_{ij})$ with $A = \bar{A}^T$ and $Q = \sum a_{ij}\bar{\eta}_i\eta_j$. The signature σ of Q is $\sigma_+ - \sigma_-$, where for some P, $\bar{P}^T A P = \mathrm{diag}(1,\overset{\sigma_+}{\dots},1,-1,\overset{\sigma_-}{\dots},-1,0,\dots)$.

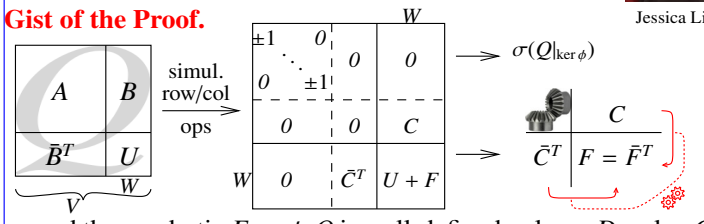
A *Partial Quadratic (PQ)* on V is a quadratic Q defined only on a subspace $\mathcal{D}_Q \subset V$. We add PQs with $\mathcal{D}_{Q_1+Q_2} := \mathcal{D}_{Q_1} \cap \mathcal{D}_{Q_2}$. Given a linear $\psi\colon V \to W$ and a PQ Q on W, there is an obvious *pullback* ψ^*Q, a PQ on V.

Jessica Liu

Theorem 1 (with Jessica Liu). Given a linear $\phi\colon V \to W$ and a PQ Q on V, there is a unique *pushforward* PQ ϕ_*Q on W such that for every PQ U on W,

$$\sigma_V(Q + \phi^*U) = \sigma_{\ker\phi}(Q|_{\ker\phi}) + \sigma_W(U + \phi_*Q).$$

Gist of the Proof.



…and the quadratic $F =: \phi_*Q$ is well-defined only on $D := \ker C$.

(more at ωεβ/icerm.)

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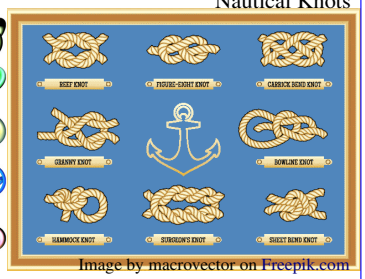
Knots and Tangles.

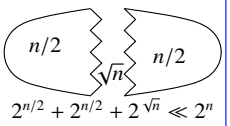


"Nautical Knots"

Image by macrovector on Freepik.com

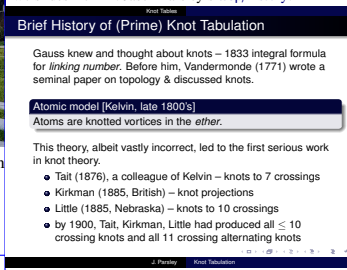
Why Tangles? • As common as knots!

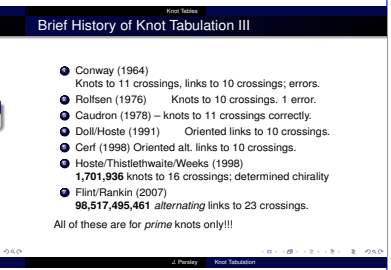
- Faster computations!
- Conceptually clearer proofs of invariance (and of skein relations).
- Often fun and consequential:
  - The Alexander polynomial ⇝ Zombian = det.
  - Knot signatures ⇝ Pushforwards of quadratic forms.
  - The Jones Polynomial ⇝ The Temperley-Lieb Algebra.
  - Khovanov Homology ⇝ "Unfinished complexes", complexes in a category.
  - The Kontsevich Integral ⇝ Drinfel'd Associators. …



One more story is left to tell, of knot tabulation.

Two slides from R. Jason Parsley's ωεβ/history:





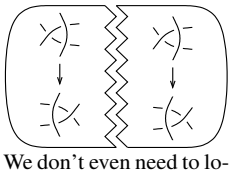
There's also Burton's tabulation to 19 crossings ωεβ/Burton, and Khesin's K250, arXiv:1705.10319.

Embarrassment 1 (personal). I don't know how to reproduce the Rolfsen table of knots! Many others can, yet I still take it on faith, contradicting one of the tenets of our practice, "thou shalt not use what thou canst not prove".

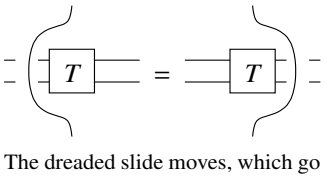
It's harder than it seems! Producing all knot diagrams is a mess, identifying all available Reidemeister moves is a mess, and you sometimes have to go up in crossing number before you can go down again.

Embarrassment 2 (communal). There isn't anywhere a tabulation of tangles! When you want to test your new discoveries, where do you go?

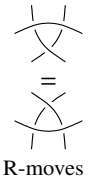
Dream. Conquer both embarrassments at once. Reproduce the Rolfsen table, and extend it to tangles, using code of the highest level of beauty. The algorithm should be so clear and simple that anyone should be able to easily implement it in an afternoon without messing with any technicalities.



We don't even need to look at all knot diagrams!



The dreaded slide moves, which go up in crossing number, are parametrized by tangles!



R-moves are tangle equalities!