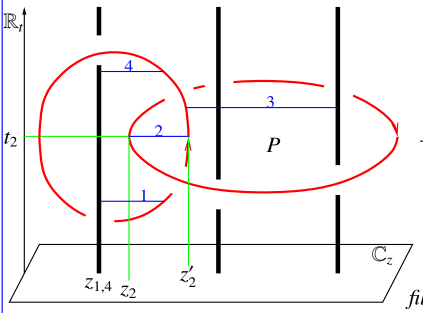
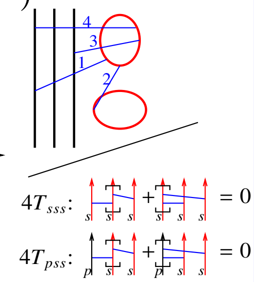
## Kontsevich in a Pole Dance Studio. (w/o poles? See [Ko, BN])

$$Z = \left( \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \sum_{\substack{t_1<\dots<t_m \\ P=\{(z_i,z_i')\}}} (-1)^{\#P_\downarrow} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \right)^{\sim} \in \mathcal{A}$$



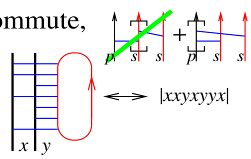

*graded by the number of chords*
*filtered by the number of ss chords*

## Comments on the Kontsevich Integral.

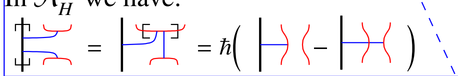
1. In the tangle case, the endpoints are fixed at top and bottom.
2. The $(\cdots)^{\sim}$ means "a correction is needed near the caps and the cups" (for the framed version, see [LM2, Da]).
3. There are never $pp$ chords, and no $4T_{pps}$ and $4T_{ppp}$ relations.
4. $Z$ is an "expansion".
5. $Z$ respects the $ss$ filtration and so descends to $Z^{/s}\colon \mathcal{K}^{/s} \to \mathcal{A}^{/s}$.

## Comments on $\mathcal{A}$.

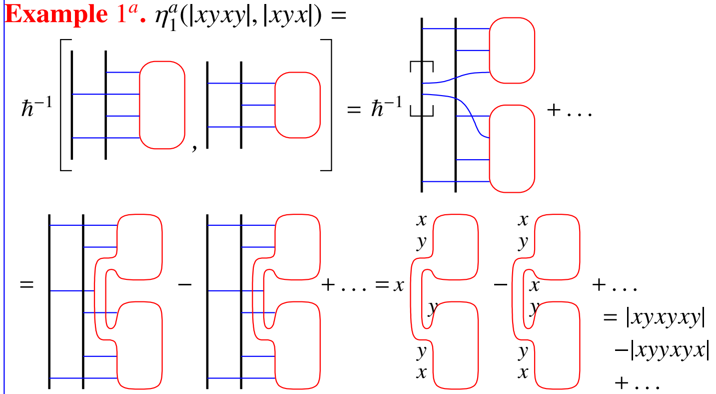
In $\mathcal{A}^{/1}$ legs on poles commute, so $\mathcal{A}^{/1}(\bigcirc) = |A|$!



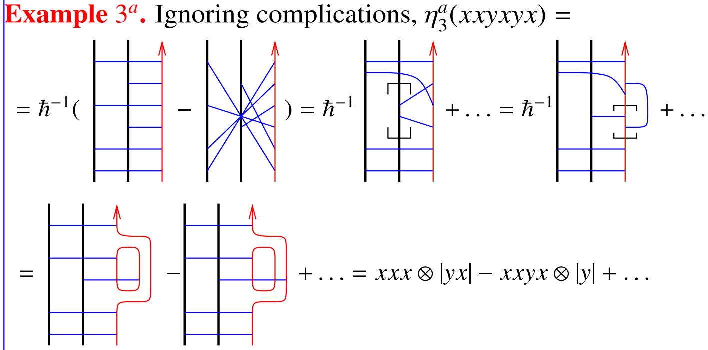

In $\mathcal{A}_H^{/2}$ we have:




## Example $1^a$. $\eta_1^a(|xyxy|, |xyx|) =$

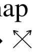
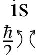
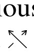
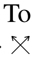
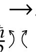



## Example $3^a$. Ignoring complications, $\eta_3^a(xxyxyx) =$




**Proof of Lemma 1.** We partially prove Theorem 2 instead:

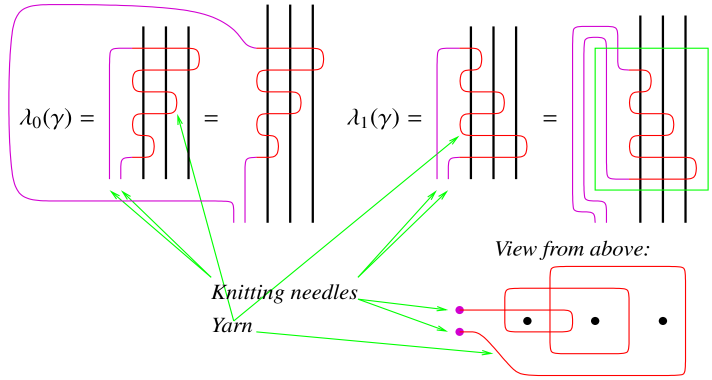
**Theorem 2.** $\operatorname{gr}^{\bullet} \mathcal{K}_H \cong \mathbb{F}[\![\hbar]\!] \otimes (\mathcal{K}^{/1})_0$.

**Proof mod $\hbar^2$.** The map $\leftarrow$ is obvious. To go $\to$, map $\mathcal{K}_H \to \mathbb{F}[\![\hbar]\!] \otimes \mathcal{K}^{/1}$ using  $\mapsto$  $+ \frac{\hbar}{2}$  and  $\mapsto$  $- \frac{\hbar}{2}$  and apply the functor $\operatorname{gr}^{\bullet}$.

## Unignoring the Complications.

We need $\lambda_0$ and $\lambda_1$ such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. Up to conjugation, $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma)) = W(\gamma)$.




## Homework

1. Is there more than Examples 1–4?
2. Derive the bialgebra axioms from this perspective.
3. What more do we get if we don't mod out by HOMFLY-PT?
4. What more do we get if we allow more than one strand-strand interaction?
5. In this language, recover Kashiwara-Vergne [AKKN1, AKKN2].
6. How is all this related to w-knots?
7. Do the same with associators. Use that to derive formulas for solutions of Kashiwara-Vergne.
8. What's the relationship with the Habiro-Massuyeau invariants of links in handlebodies [HM] (different filtration!).
9. Pole dance on other surfaces!
10. Explore the action of the mapping class group.

Kashiwara Vergne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