Great Embarrassment 1. I don't know if any of the Alexander, Jones, HOMFLY-PT, and Kauffman polynomials is C3D. I don't know if any Reshetikhin-Turaev invariant is C3D. I don't know if any knot homology is C3D.

Or maybe it's a cause for optimism — there's still something very basic we don't know about (say) the Jones polynomial. Can we understand it well enough 3-dimensionally to compute it well? If not, why not?

Conversation Starter 2. Similarly, if η is a stingy quantity (a quantity we expect to be small for small knots), we will say that η has Savings in 3D, or "has S3D" if $M_\eta(3D, V) \ll M_\eta(2D, V^{4/3})$.

Example (R. van der Veen, D. Thurston, private communications). The hyperbolic volume has *S3D*.

Great Embarrassment 2. I don't know if the genus of a knot has S3D! In other words, even if a knot is given in a 3-dimensional, the best way I know to find a Seifert surface for it is to first project it to 2D, at a great cost.

Next we argue that most finite type invariants are probably C3D...

(What a weak statement!)

Theorem FT2D. If ζ is a finite type invariant of type d then $C_\zeta(2D, n)$ is at most $\sim n^{\lfloor 3d/4 \rfloor}$. With more effort, $C_\zeta(2D, n) \lesssim n^{(\frac{2}{3}+\epsilon)d}$.

Note that there are some exceptional finite type invariants, e.g. high coefficients of the Alexander polynomial and other poly-time knot polynomials, which can be computed much faster!

Theorem FT3D. If ζ is a finite type invariant of type d then $C_\zeta(3D, V)$ is at most $\sim V^{6d/7+1/7}$. With more effort, $C_\zeta(2D, V) \lesssim V^{(\frac{4}{5}+\epsilon)d}$.

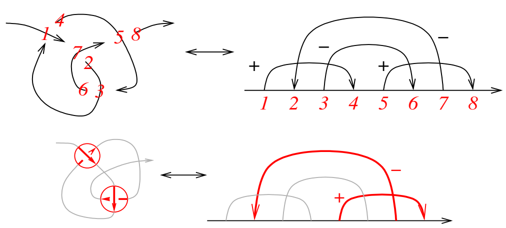
All pre-categorification knot polynomials are power series whose coefficients are finite type invariants. (This is sometimes helpful for the computation of finite type invariants, but rarely helpful for the computation of knot polynomials).

Tentative Conclusion. As

$n^{3d/4} \sim (V^{4/3})^{3d/4} = V \gg V^{6d/7+1/7}$ $\qquad n^{2d/3} \sim (V^{4/3})^{2d/3} = V^{8d/9} \gg V^{4d/5}$

these theorems say "most finite type invariants are probably C3D; the ones in greater doubt are the lucky few that can be computed unusually quickly".

Gauss diagrams and sub-Gauss-diagrams:

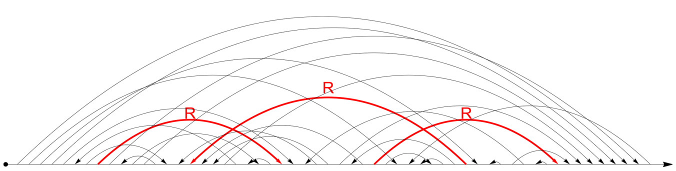


Let $\varphi_d\colon \{\text{knot diagrams}\} \to \langle\text{Gauss diagrams}\rangle$ map every knot diagram to the sum of all the sub-diagrams of its Gauss diagram which have at most d arrows.

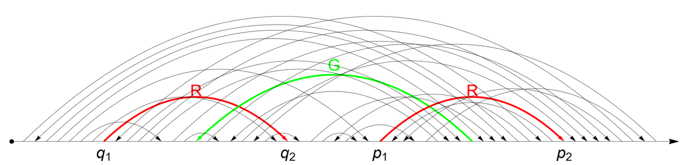
Under-Explained Theorem (Goussarov-Polyak-Viro). A knot invariant ζ is of type d iff there is a linear functional ω on $\langle\text{Gauss diagrams}\rangle$ such that $\zeta = \omega \circ \varphi_d$.

Theorem FT2D. If ζ is a finite type invariant of type d then $C_\zeta(2D, n)$ is at most $\sim n^{\lfloor 3d/4 \rfloor}$. With more effort, $C_\zeta(2D, n) \lesssim n^{(\frac{2}{3}+\epsilon)d}$.

Proof of Theorem FT2D.



We need to count how many times a diagram such as the red appears within a bigger diagram, having n arrows. Clearly this can be done in time $\sim n^3$, and in general, in time $\sim n^d$.



With an appropriate look-up table, it can also be done in time $\sim n^2$ (in general, $\sim n^{d-1}$). That look-up table $(T^{p_1,p_2}_{q_1,q_2})$ is of size (and production cost) $\sim n^4$ if you are naive, and $\sim n^2$ if you are just a bit smarter. Indeed

$$T^{p_1,p_2}_{q_1,q_2} = T^{0,p_2}_{0,q_2} - T^{0,p_1}_{0,q_2} - T^{0,p_2}_{0,q_1} + T^{0,p_1}_{0,q_1},$$

and $(T^{0,p}_{0,q})$ is easy to compute.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/KOS-211021/