Multi-feathers and multi-sharks.

For a type *d* invariant we need to count *d*-tuples of crossings, and each has its own "generation" g_i . So we have the "multi-generation"

$$\bar{g} = (g_1, \ldots, g_d)$$

Let $G := \sum g_i$ be the "overall generation". We will choose between a "multi-feather" method and a "multi-shark" method based on the size of G.



The effort to take a single multi-bite is tiny. Indeed, **Lemma** Given 2d finite sets $B_i = \{t_{i1}, t_{i2}, \ldots\} \subset [1..L^3]$ and a permutation $\pi \in S_{2n}$ the quantity

$$N = \left| \left\{ (b_i) \in \prod_{i=1}^{2d} B_i \colon \text{the } b_i \text{'s are ordered as } \pi \right\} \right|$$

can be computed in time $\sim \sum |B_i| \sim \max |B_i|$.

Proof. WLOG $\pi = Id$. For $\iota \in [1..2d]$ and $\beta \in B \coloneqq \cup B_i$ let

$$N_{\iota,\beta} = \left| \left\{ (b_i) \in \prod_{i=1}^{\iota} B_i \colon b_1 < b_2 < \ldots < b_{\iota} \le \beta \right\} \right|.$$

We need to know $N_{2d,\max B}$; compute it inductively using $N_{\iota,\beta} = N_{\iota,\beta'} + N_{\iota-1,\beta'}$, where β' is the predecessor of β in B.







Conclusion. We wish to compute the contribution to φ_d coming from *d*-tuples of crossings of multi-generation \bar{g} .

► The multi-shark method does it in time

$$\sim$$
 (no. of bites) \cdot (time per bite) = $L^{2d}2^{G} \cdot \frac{L}{2^{\min \tilde{g}}} < L^{2d+1}2^{G}$

(increases with G).

> The multi-feather method (project and use the 2D algorithm) does it in time

$$\sim (\text{no. of crossings})^{\lfloor \frac{3}{4}d \rfloor} = \left(\prod_{i=1}^{d} L^2 \frac{L^2}{2^{\mathcal{E}_i}}\right)^{\lfloor \frac{3}{4}d \rfloor} < \frac{L^{3d}}{(2^G)^{3/4}}$$

(decreases with G).

Of course, for any specific ${\cal G}$ we are free to choose whichever is better, shark or feather.

The two methods agree (and therefore are at their worst) if $2^G = L^{\frac{4}{7}(d-1)}$, and in that case, they both take time $\sim L^{\frac{18}{7}d+\frac{3}{7}} = V^{\frac{6}{7}d+\frac{1}{7}}$.

The same reasoning, with the $n^{(rac{2}{3}+\epsilon)d}$ feather, gives $V^{(rac{4}{5}+\epsilon)d}$

If time — a word about braids.

Thank You!