

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Convention. For a finite set A , let $z_A := \{z_i\}_{i \in A}$ and let $\zeta_A := \{\zeta_i^* = \zeta_i\}_{i \in A}$.
 $(p, x)^* = (\pi, \xi)$

The Generating Series \mathcal{G} : $\text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B]) \rightarrow \mathbb{Q}[[\zeta_A, z_B]]$.

Claim. $L \in \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B]) \xrightarrow{\mathcal{G}} \mathbb{Q}[z_B][[\zeta_A]] \ni L$ via

$$\mathcal{G}(L) := \sum_{n \in \mathbb{N}^A} \frac{\zeta_A^n}{n!} L(z_A^n) = L\left(\bigoplus_{a \in A} \zeta_a z_a\right) = L = \text{greek } \mathcal{L} \text{ latin},$$

$$G^{-1}(L)(p) = \left(p|_{z_a \rightarrow \partial_{\zeta_a}} L\right)_{\zeta_a=0} \quad \text{for } p \in \mathbb{Q}[z_A].$$

Claim. If $L \in \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B])$, $M \in \text{Hom}(\mathbb{Q}[z_B] \rightarrow \mathbb{Q}[z_C])$, then $\mathcal{G}(L/M) = (\mathcal{G}(L)|_{z_b \rightarrow \partial_{\zeta_b}} \mathcal{G}(M))_{\zeta_b=0}$.

Examples. • $G(id: \mathbb{Q}[p, x] \rightarrow \mathbb{Q}[p, x]) = \mathbb{Q}^{p+\xi x}$.

• Consider $R_{ij} \in (\mathfrak{h}_i \otimes \mathfrak{h}_j)[[t]] \cong \text{Hom}(\mathbb{Q}[] \rightarrow \mathbb{Q}[p_i, x_i, p_j, x_j])[[t]]$. Then $\mathcal{G}(R_{ij}) = \mathbb{Q}^{(e^t-1)(p_i-p_j)x_j} = \mathbb{Q}^{(T-1)(p_i-p_j)x_j}$.

Heisenberg Algebras. Let $\mathfrak{h} = A\langle p, x \rangle / ([p, x] = 1)$, let $\mathbb{O}_i: \mathbb{Q}[p_i, x_i] \rightarrow \mathfrak{h}_i$ is the “ p before x ” PBW normal ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[p_i, x_i, p_j, x_j] \xrightarrow{\mathbb{O}_i \otimes \mathbb{O}_j} \mathfrak{h}_i \otimes \mathfrak{h}_j \xrightarrow{m_k^{ij}} \mathfrak{h}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[p_k, x_k].$$

Then $\mathcal{G}(hm_k^{ij}) = \mathbb{Q}^{-\xi_i \pi_j + (\pi_i + \pi_j)p_k + (\xi_i + \xi_j)x_k}$.

Proof. Recall the “Weyl CCR” $\mathbb{Q}^{\xi_i x_i} \mathbb{Q}^{\pi p} = \mathbb{Q}^{-\xi_i \pi} \mathbb{Q}^{\pi p} \mathbb{Q}^{\xi x}$, and find

$$\begin{aligned} \mathcal{G}(hm_k^{ij}) &= \mathbb{Q}^{\pi_i p_i + \xi_i x_i + \pi_j p_j + \xi_j x_j} // \mathbb{O}_i \otimes \mathbb{O}_j // m_k^{ij} // \mathbb{O}_k^{-1} \\ &= \mathbb{Q}^{\pi_i p_i + \xi_i x_i + \pi_j p_j + \xi_j x_j} // m_k^{ij} // \mathbb{O}_k^{-1} = \mathbb{Q}^{\pi_i p_k + \xi_i x_k + \pi_j p_k + \xi_j x_k} // \mathbb{O}_k^{-1} \\ &= \mathbb{Q}^{-\xi_i \pi_j + (\pi_i + \pi_j)p_k + (\xi_i + \xi_j)x_k} // \mathbb{O}_k^{-1} = \mathbb{Q}^{-\xi_i \pi_j + (\pi_i + \pi_j)p_k + (\xi_i + \xi_j)x_k}. \end{aligned}$$

GDO := The category with objects finite sets and

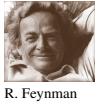
$$\text{mor}(A \rightarrow B) = \{\mathcal{L} = \omega \mathbb{Q}\} \subset \mathbb{Q}[[\zeta_A, z_B]].$$

where: • ω is a scalar. • Q is a “small” quadratic in $\zeta_A \cup z_B$.

• Compositions: $\mathcal{L}/\mathcal{M} := (\mathcal{L}|_{z_i \rightarrow \partial_{\zeta_i}} \mathcal{M})_{\zeta_i=0}$.

Compositions. In $\text{mor}(A \rightarrow B)$,

$$Q = \sum_{i \in A, j \in B} E_{ij} \zeta_i z_j + \frac{1}{2} \sum_{i, j \in A} F_{ij} \zeta_i \zeta_j + \frac{1}{2} \sum_{i, j \in B} G_{ij} z_i z_j,$$



and so

(remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)

$$\begin{array}{c} A: \omega_1 \\ \hline E_1 \\ Q_1 \\ F_1 \quad G_1 \end{array} \text{ greek} \quad \text{and} \quad \begin{array}{c} B: \omega_2 \\ \hline E_2 \\ Q_2 \\ F_2 \quad G_2 \end{array} \text{ latin} \quad \text{and} \quad \begin{array}{c} A: \omega \\ \hline E \\ Q \\ F \quad G \end{array} \text{ greek} \quad \text{and} \quad \begin{array}{c} C: \\ \hline E_1 E_2 + E_1 F_2 G_1 E_2 \\ + E_1 F_2 G_1 F_2 G_1 E_2 \\ + \dots \\ = \sum_{r=0}^{\infty} E_1 (F_2 G_1)^r E_2 \end{array}$$

where • $E = E_1(I - F_2 G_1)^{-1} E_2$ • $F = F_1 + E_1 F_2 (I - G_1 F_2)^{-1} E_2$

• $G = G_2 + E_2^T G_1 (I - F_2 G_1)^{-1} E_2$ • $\omega = \omega_1 \omega_2 \det(I - F_2 G_1)^{-1/2}$

Proof of Claim in Example 2. Let $\Phi_1 := \mathbb{Q}^{t(p_i-p_j)x_j}$ and $\Phi_2 := \mathbb{Q}_{p_j x_j} (\mathbb{Q}^{(e^t-1)(p_i-p_j)x_j}) =: \mathbb{Q}(\Psi)$. We show that $\Phi_1 = \Phi_2$ in $(\mathfrak{h}_i \otimes \mathfrak{h}_j)[[t]]$ by showing that both solve the ODE $\partial_t \Phi = (p_i - p_j)x_j \Phi$ with $\Phi|_{t=0} = 1$. For Φ_1 this is trivial. $\Phi_2|_{t=0} = 1$ is trivial, and

$$\partial_t \Phi_2 = \mathbb{Q}(\partial_t \Psi) = \mathbb{Q}(\mathbb{Q}^t(p_i - p_j)x_j \Psi)$$

$$(p_i - p_j)x_j \Phi_2 = (p_i - p_j)x_j \mathbb{Q}(\Psi) = (p_i - p_j)\mathbb{Q}(x_j \Psi - \partial_{p_j} \Psi)$$

$$= \mathbb{Q}((p_i - p_j)(x_j \Psi + (\mathbb{Q}^t - 1)x_j \Psi)) = \mathbb{Q}(\mathbb{Q}^t(p_i - p_j)x_j \Psi) \quad \square$$

Implementation.

Without, don't trust!

$\text{CF} = \text{ExpandNumerator} @ * \text{ExpandDenominator} @ * \text{PowerExpand} @ * \text{Factor};$

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 $\text{E}_{A1 \rightarrow B1}[\omega_1, Q1] \text{E}_{A2 \rightarrow B2}[\omega_2, Q2] := \text{E}_{A1 \cup A2 \rightarrow B1 \cup B2}[\omega_1 \omega_2, Q1 + Q2]$ 
 $(\text{E}_{A1 \rightarrow B1}[\omega_1, Q1] // \text{E}_{A2 \rightarrow B2}[\omega_2, Q2]) /; (B1^* == A2) :=$ 
 $\text{Module}[\{i, j, E1, F1, G1, E2, F2, G2, I, M = \text{Table}\},$ 
 $I = \text{IdentityMatrix} @ \text{Length}@B1;$ 
 $E1 = M[\partial_{i,j} Q1, \{i, A1\}, \{j, B1\}]; E2 = M[\partial_{i,j} Q2, \{i, A2\}, \{j, B2\}];$ 
 $F1 = M[\partial_{i,j} Q1, \{i, A1\}, \{j, A1\}]; F2 = M[\partial_{i,j} Q2, \{i, A2\}, \{j, A2\}];$ 
 $G1 = M[\partial_{i,j} Q1, \{i, B1\}, \{j, B1\}]; G2 = M[\partial_{i,j} Q2, \{i, B2\}, \{j, B2\}];$ 
 $\text{E}_{A1 \rightarrow B2}[\text{CF}[\omega_1 \omega_2 \text{Det}[I - F2.G1]^{1/2}], \text{CF}@\text{Plus}[$ 
 $\text{If}[A1 == \{\} \vee B2 == \{\}, 0, A1.E1.\text{Inverse}[I - F2.G1].E2.B2],$ 
 $\text{If}[A1 == \{\}, 0, \frac{1}{2} A1.(F1 + E1.F2.\text{Inverse}[I - G1.F2].E1).A1],$ 
 $\text{If}[B2 == \{\}, 0, \frac{1}{2} B2.(G2 + E2'.G1.\text{Inverse}[I - F2.G1].E2).B2]]]]$ 
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$A \setminus B := \text{Complement}[A, B];$

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 $(\text{E}_{A1 \rightarrow B1}[\omega_1, Q1] // \text{E}_{A2 \rightarrow B2}[\omega_2, Q2]) /; (B1^* != A2) :=$ 
 $\text{E}_{A1 \cup (A2 \setminus B1) \rightarrow B1 \cup A2^*}[\omega_1, Q1 + \text{Sum}[S^*, \zeta, \{\zeta, A2 \setminus B1^*\}]] //$ 
 $\text{E}_{B1 \star A2 \rightarrow B2 \cup (B1 \setminus A2^*)}[\omega_2, Q2 + \text{Sum}[Z^*, z, \{z, B1 \setminus A2^*\}]]$ 
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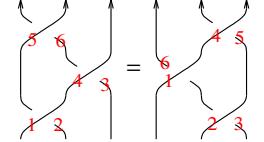
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 $\{p^*, x^*, \pi^*, \xi^*\} = \{\pi, \xi, p, x\}; (u_i)^* := (u^*)_i;$ 
 $L\_List^* := \#^* \& @/ L;$ 
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 $R_{i_, j_} := \text{E}_{()} \rightarrow \{p_i, x_i, p_j, x_j\} [T^{-1/2}, (1 - T) p_j x_j + (T - 1) p_i x_j];$ 
 $\bar{R}_{i_, j_} := \text{E}_{()} \rightarrow \{p_i, x_i, p_j, x_j\} [T^{1/2}, (1 - T^{-1}) p_j x_j + (T^{-1} - 1) p_i x_j];$ 
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 $C_{i_} := \text{E}_{()} \rightarrow \{p_i, x_i\} [T^{1/2}, 0];$ 
 $\bar{C}_{i_} := \text{E}_{()} \rightarrow \{p_i, x_i\} [T^{1/2}, 0];$ 
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 $h_{i_, j_ \rightarrow k_} := \text{E}_{\{\pi_i, \xi_i, \pi_j, \xi_j\} \rightarrow \{p_k, x_k\}} [1, -\xi_i \pi_j + (\pi_i + \pi_j) p_k + (\xi_i + \xi_j) x_k]$ 
```

```
 $\text{E}_{()} \rightarrow \{vs, Q\}_h := \text{Module}[\{ps, xs, M\},$ 
 $ps = \text{Cases}[vs, p_]; xs = \text{Cases}[vs, x_];$ 
 $M = \text{Table}[\omega_i, 1 + \text{Length}@ps, 1 + \text{Length}@xs];$ 
 $M[1;; 2;; 2;;] = \text{Table}[\text{CF}[\partial_{i,j} Q], \{i, ps\}, \{j, xs\}];$ 
 $M[2;; 1;;] = ps; M[1;; 2;;] = xs;$ 
 $\text{MatrixForm}[M]_h]$ 
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Proof of Reidemeister 3.

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 $(R_{1,2} R_{4,3} R_{5,6} // hm_{1,4 \rightarrow 2} hm_{2,5 \rightarrow 2} hm_{3,6 \rightarrow 3}) ==$ 
 $(R_{2,3} R_{1,6} R_{4,5} // hm_{1,4 \rightarrow 1} hm_{2,5 \rightarrow 2} hm_{3,6 \rightarrow 3})$ 
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True

□

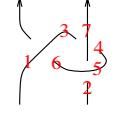
The “First Tangle”.

Factor /@

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(z = R_{1,2} R_{4,3} R_{5,6} // hm_{1,4 \rightarrow 2} hm_{2,5 \rightarrow 2} hm_{3,6 \rightarrow 3}) ==
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(E_{()} \rightarrow \{p1, p2, x1, x2\} [-1 + 2 T, (-1 + T) (p1 - p2) (T x1 - x2)] /-1 + 2 T)
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$$z_h = \begin{pmatrix} \frac{-1+2 T}{T} & x_1 & x_2 \\ p_1 & \frac{-T+T^2}{-1+2 T} & \frac{1-T}{-1+2 T} \\ p_2 & \frac{T-T^2}{-1+2 T} & \frac{-1+T}{-1+2 T} \end{pmatrix}_h$$

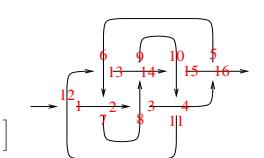


The knot 8_17.

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z = R_{12,1} R_{27} R_{33} R_{4,11} R_{16,5} R_{6,13} R_{14,9} R_{10,15};
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Table[z = z // hm_{ik \rightarrow 1}, \{k, 2, 16\}] // Last
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E_{()} \rightarrow \{p1, x1\} [1 - 4 T + 8 T^2 - 11 T^3 + 8 T^4 - 4 T^5 + T^6, 0]
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Proof of Theorem 3, (3).

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{Y1 = E_{()} \rightarrow \{p1, x1, p2, x2, p3, x3\} [\omega, \{p1, p2, p3\}. \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \varphi \end{pmatrix} . \{x1, x2, x3\}]}
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(Y1 // hm_{1,2 \rightarrow \theta})_h}
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$$\left\{ \begin{pmatrix} \omega & x_1 & x_2 & x_3 \\ p_1 & \alpha & \beta & \theta \\ p_2 & \gamma & \delta & \epsilon \\ p_3 & \phi & \psi & \varphi \end{pmatrix}_h, \begin{pmatrix} \omega + \gamma \omega & x_\theta & x_3 \\ p_\theta & \frac{\alpha + \beta + \gamma + \delta - \alpha \delta}{1 + \gamma} & \frac{\epsilon - \alpha + \theta + \gamma \theta}{1 + \gamma} \\ p_3 & \frac{\phi - \delta \phi + \beta + \gamma \psi}{1 + \gamma} & \frac{\frac{1 + \gamma}{\gamma + \varphi - \epsilon - \phi}}{1 + \gamma} \end{pmatrix}_h \right\}$$

□

References.

On $\omega\beta\beta=\text{http://drorbn.net/cat20}$