Dror Bar-Natan: Talks: Toronto-1912:

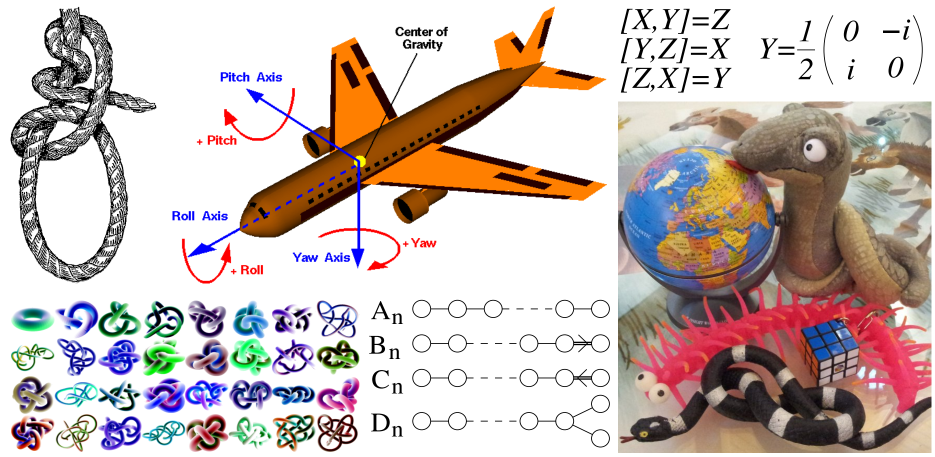
Chord Diagrams, Knots, and Lie Algebras

Thanks for inviting me to the *Chord Diagrams Everywhere* session / Winter 2019 CMS meeting! ωεβ:=http://drorbn.net/to19

Abstract. This will be a service talk on ancient material — I will briefly describe how the exact same type of chord diagrams (and relations between them) occur in a natural way in both knot theory and in the theory of Lie algebras.

While preparing for this talk I realized that I've done it before, much better, within a book review. So here's that review! It has been modified from its original version: it had been formatted to fit this page, parts were highlighted, and commentary had been added in green italics.

[Book] *Introduction to Vassiliev Knot Invariants,* by S. Chmutov, S. Duzhin, and J. Mostovoy, Cambridge University Press, Cambridge UK, 2012, xvi+504 pp., hardback, \$70.00, ISBN 978-1-10702-083-2.



A knot and a Lie algebra, a list of knots and a list of Lie algebras, and an unusual conference of the symmetric and the knotted.

Merely ~~30~~ *36* years ago, if you had asked even the best informed mathematician about the relationship between knots and Lie algebras, she would have laughed, for there isn't and there can't be. Knots are flexible; Lie algebras are rigid. Knots are irregular; Lie algebras are symmetric. The list of knots is a lengthy mess; the collection of Lie algebras is well-organized. Knots are useful for sailors, scouts, and hangmen; Lie algebras for navigators, engineers, and high energy physicists. Knots are blue collar; Lie algebras are white. They are as similar as worms and crystals: both well-studied, but hardly ever together.

Then in the 1980s came Jones, and Witten, and Reshetikhin and Turaev [Jo, Wi, RT] and showed that if you really are the best informed, and you know your quantum field theory and conformal field theory and quantum groups, then you know that the two disjoint fields are in fact intricately related. This "quantum" approach remains the most powerful way to get computable knot invariants out of (certain) Lie algebras (and representations thereof). Yet shortly later, in the late 80s and early 90s, an alternative perspective arose, that of "finite-type" or "Vassiliev-Goussarov" invariants [Va1, Va2, Go1, Go2, BL, Ko1, Ko2, BN1], which made the surprising relationship between knots and Lie algebras appear simple and almost inevitable.

The reviewed [Book] is about that alternative perspective, the one reasonable sounding but not entirely trivial theorem that is crucially needed within it (the "Fundamental Theorem" or the "Kontsevich integral"), and the many threads that begin with that perspective. Let me start with a brief summary of the mathematics, and even before, an even briefer summary.

In briefest, a certain space $\mathcal{A}$ of chord diagrams is the dual to the dual of the space of knots, and at the same time, it is dual to Lie algebras.

The briefer summary is that in some combinatorial sense it is possible to "differentiate" knot invariants, and hence it makes sense to talk about "polynomials" on the space of knots — these are functions on the set of knots (namely, these are knot invariants) whose sufficiently high derivatives vanish. Such polynomials can be fairly conjectured to separate knots — elsewhere in math in lucky cases polynomials separate points, and in our case, specific computations are encouraging. Also, such polynomials are determined by their "coefficients", and each of these, by the one-side-easy "Fundamental Theorem", is a linear functional on some finite space of

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Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/