graphs modulo relations. These same graphs turn out to parameterize formulas that make sense in a wide class of Lie algebras, and the said relations match exactly with the relations in the definition of a Lie algebra — anti-symmetry and the Jacobi identity. Hence what is more or less dual to knots (invariants), is also, after passing to the coefficients, dual to certain graphs which are more or less dual to Lie algebras. QED, and on to the less brief summary[1].

Let V be an arbitrary invariant of oriented knots in oriented space with values in (say) $\mathbb{Q}$. Extend V to be an invariant of 1-singular knots, knots that have a single singularity that locally looks like a double point $\times$, using the formula

$$V(\times) = V(\nearrow\!\!\!\nwarrow) - V(\nwarrow\!\!\!\nearrow). \qquad (1)$$

Further extend V to the set $\mathcal{K}^m$ of m-singular knots (knots with m such double points) by repeatedly using (1).

Definition 1. We say that V is of type m (or "Vassiliev of type m") if its extension $V|_{\mathcal{K}^{m+1}}$ to $(m+1)$-singular knots vanishes identically. We say that V is of finite type (or "Vassiliev") if it is of type m for some m.

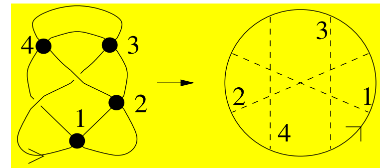
Repeated differences are similar to repeated derivatives and hence it is fair to think of the definition of $V|_{\mathcal{K}^m}$ as repeated differentiation. With this in mind, the above definition imitates the definition of polynomials of degree m. Hence finite type invariants can be thought of as "polynomials" on the space of knots[2]. It is known (see e.g. [Book]) that the class of finite type invariants is large and powerful. Yet the first question on finite type invariants remains unanswered:

Problem 2. *Honest polynomials are dense in the space of functions. Are finite type invariants dense within the space of all knot invariants? Do they separate knots?*

The top derivatives of a multi-variable polynomial form a system of constants that determine that polynomial up to polynomials of lower degree. Likewise the mth derivative[3] $V^{(m)} = V|_{\mathcal{K}^m} = V\left(\times\overset{m}{\cdots}\times\right)$ of a type m invariant V is a constant in the sense that it does not see the difference between overcrossings and undercrossings and so it is blind to 3D topology. Indeed

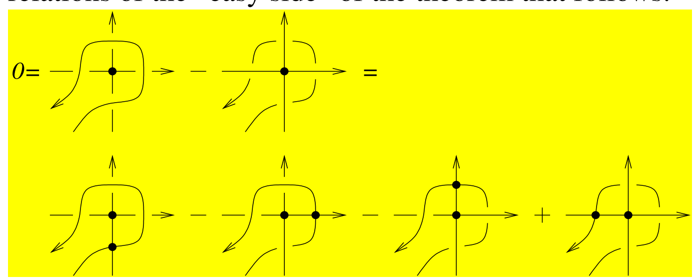
$$V\left(\times\overset{m}{\cdots}\times\nearrow\!\!\!\nwarrow\right) - V\left(\times\overset{m}{\cdots}\times\nwarrow\!\!\!\nearrow\right) = V\left(\times\overset{m+1}{\cdots}\times\right) = 0.$$

Also, clearly $V^{(m)}$ determines V up to invariants of lower type. Hence a primary tool in the study of finite type invariants is the study of the "top derivative" $V^{(m)}$, also known as "the weight system of V".



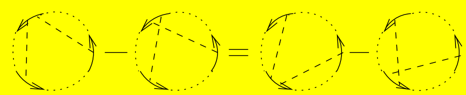
Blind to 3D topology, $V^{(m)}$ only sees the combinatorics of the circle that parameterizes an m-singular knot. On this circle there are m pairs of points that are pairwise identified in the image; standardly one indicates those by drawing a circle with m chords marked (an "m-chord diagram") as above. Let $\mathcal{D}_m$ denote the space of all formal linear combinations with rational coefficients of m-chord diagrams. Thus $V^{(m)}$ is a linear functional on $\mathcal{D}_m$.

I leave it for the reader to figure out or read in [Book, pp. 88] how the following figure easily implies the "4T" relations of the "easy side" of the theorem that follows:



Theorem 3. *(The Fundamental Theorem, details in [Book]).*

- *(Easy side) If V is a rational valued type m invariant then $V^{(m)}$ satisfies the "4T" relations shown above, and hence it descends to a linear functional on $\mathcal{A}_m := \mathcal{D}_m/4T$. If in addition $V^{(m)} \equiv 0$, then V is of type $m-1$.*



- *(Hard side, slightly misstated by avoiding "framings") For any linear functional W on $\mathcal{A}_m$ there is a rational valued type m invariant V so that $V^{(m)} = W$.*

Thus to a large extent the study of finite type invariants is reduced to the finite (though super-exponential in m) algebraic study of $\mathcal{A}_m$.

Much of the richness of finite type invariants stems from their relationship with Lie algebras. Theorem 4 below suggests this relationship on an abstract level and Theorem 5 makes that relationship concrete.

[1] Partially self-plagiarized from [BN2].

[2] Keep this apart from invariants of knots whose values are polynomials, such as the Alexander or the Jones polynomial. A posteriori related, these are a priori entirely different.

[3] As common in the knot theory literature, in the formulas that follow a picture such as $\times\overset{m}{\cdots}\times\nearrow\!\!\!\nwarrow$ indicates "some knot having m double points and a further (right-handed) crossing". Furthermore, when two such pictures appear within the same formula, it is to be understood that the parts of the knots (or diagrams) involved *outside* of the displayed pictures are to be taken as the same.