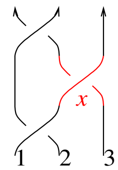
Dror Bar-Natan: Talks: Toronto-1912: ωεβ:=http://drorbn.net/to19/

Geography vs. Identity

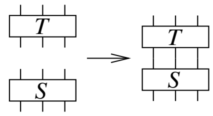
Thanks for inviting me to the *Topology* session!

Abstract. Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

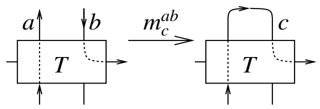


Geography view: $\gamma_1 = $ ⤫ | | $\gamma_2 = $ | ⤫ | $\gamma_3 = $ | | ⤫ … so x is γ_2.

Identity view: At x strand 1 crosses strand 3, so x is σ_{13}.



Geographers care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary "stacking of tangles". They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call "quantum topology".



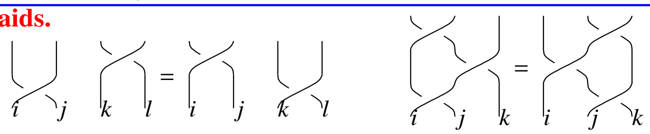
Identiters believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation m^{ab}_c, and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See ωεβ/reg, ωεβ/kbh.

The Gold Standard is set by the "Γ-calculus" Alexander formulas (ωεβ/mac). An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\}$ with $R_S := \mathbb{Z}(\{T_a : a \in S\})$:

$$\left({}_a\!\nearrow\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\nearrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-T_a^{\pm 1} \\ b & 0 & T_a^{\pm 1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[T_a, T_b \to T_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array} \right)$$

Braids.



Geography: (better topology!)

$$GB := \langle \gamma_i \rangle \Big/ \left(\begin{array}{c} \gamma_i\gamma_k = \gamma_k\gamma_i \text{ when } |i-k|>1 \\ \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1} \end{array} \right) = B.$$

Identity: (captures quantum algebra!)

$$IB := \langle \sigma_{ij} \rangle \Big/ \left(\begin{array}{c} \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \text{ when } |\{i,j,k,l\}| = 4 \\ \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \text{ when } |\{i,j,k\}| = 3 \end{array} \right) = PvB.$$

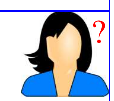
Theorem. Let $S = \{\tau\}$ be the symmetric group. Then vB is both

$$PvB \rtimes S \cong B * S \Big/ \left(\gamma_i\tau = \tau\gamma_j \text{ when } \tau i = j,\ \tau(i+1) = (j+1) \right)$$

(and so PvB is "bigger" then B, and hence quantum algebra doesn't see topology very well).

Proof. Going left, $\gamma_i \mapsto \sigma_{i,i+1}(i\ i+1)$. Going right, if $i<j$ map $\sigma_{ij} \mapsto (j{-}1\ j{-}2\ \ldots\ i)\gamma_{j-1}(i\ i{+}1\ \ldots\ j)$ and if $i>j$ use $\sigma_{ij} \mapsto (j\ j{+}1\ \ldots\ i)\gamma_j(i\ i{-}1\ \ldots\ j{+}1)$.

vB views of σ_{ij}:



Betty Jane Gassner deserves to be more famous

The Gassner Representation of PvB_n acts on $V = R^n := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^n = R\langle v_1, \ldots, v_n\rangle$ by

$$\sigma_{ij} v_k = v_k + \delta_{kj}(t_i - 1)(v_j - v_i).$$

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G_{i_,j_}[ℰ_] := ℰ /. v_{k_} :→ v_k + δ_{k,j} (t_i - 1) (v_j - v_i) // Expand
(bas3 // G_{1,2} // G_{1,3} // G_{2,3}) == (bas3 // G_{2,3} // G_{1,3} // G_{1,2})
True
```

S_n acts on R^n by permuting the v_i *and* the t_i, so the Gassner representation extends to vB_n and then restricts to B_n as a $\mathbb{Z}$-linear ∞-dimensional representation. It then descends to PB_n as a finite-rank R-linear representation, with lengthy non-local formulas.

Geographers: Gassner is an obscure partial extension of Burau.

Identiters: Burau is a trivial silly reduction of Gassner.

With Roland van der Veen

The Turbo-Gassner Representation. With the same R and V, TG acts on $V \oplus (R^n \otimes V) \oplus (S^2V \otimes V^*) = R\langle v_k, v_{lk}, u_iu_jw_k\rangle$ by

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TG_{i_,j_}[ℰ_] := ℰ /. {
   v_{k_} :→ v_k + δ_{k,j} ((t_i - 1) (v_j - v_i) + v_{i,j} - v_{i,i}) +
      δ_{k,i} (u_j - u_i) u_i w_j,
   v_{l_,k_} :→ v_{l,k} + (t_i - 1) ×
      (δ_{k,j} (v_{l,j} - v_{l,i}) + (δ_{l,i} - δ_{l,j} t_i^{-1} t_j)
         (u_k + δ_{k,j} (t_i - 1) (u_j - u_i)) u_i w_j),
   u_{k_} :→ u_k + δ_{k,j} (t_i - 1) (u_j - u_i),
   w_{k_} :→ w_k + (δ_{k,j} - δ_{k,i}) (t_i^{-1} - 1) w_j} // Expand
bas3 = {v_1, v_2, v_3, v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}, v_{3,1},
   v_{3,2}, v_{3,3}, u_1^2 w_1, u_1^2 w_2, u_1^2 w_3, u_1 u_2 w_1, u_1 u_2 w_2, u_1 u_2 w_3,
   u_1 u_3 w_1, u_1 u_3 w_2, u_1 u_3 w_3, u_2^2 w_1, u_2^2 w_2, u_2^2 w_3, u_2 u_3 w_1,
   u_2 u_3 w_2, u_2 u_3 w_3, u_3^2 w_1, u_3^2 w_2, u_3^2 w_3};
(bas3 // TG_{1,2} // TG_{1,3} // TG_{2,3}) == (bas3 // TG_{2,3} // TG_{1,3} // TG_{1,2})
True
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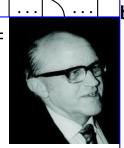
Gassner motifs

Adjoint-Gassner

Like Gassner, TG is also a representation of PB_n.

I have no idea where it belongs!



Werner Burau

The Burau Representation of PvB_n acts on $R^n := \mathbb{Z}[t^{\pm 1}]^n = R\langle v_1, \ldots, v_n\rangle$ by

$$\sigma_{ij} v_k = v_k + \delta_{kj}(t-1)(v_j - v_i).$$

ωεβ/code

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δ /: δ_{i_,j_} := If[i == j, 1, 0];
B_{i_,j_}[ℰ_] := ℰ /. v_{k_} :→ v_k + δ_{k,j} (t - 1) (v_j - v_i) // Expand
(bas3 = {v_1, v_2, v_3}) // B_{1,2}
{v_1, v_1 - t v_1 + t v_2, v_3}
bas3 // B_{1,2} // B_{1,3} // B_{2,3}
{v_1, v_1 - t v_1 + t v_2, v_1 - t v_1 + t v_2 - t^2 v_2 + t^2 v_3}
bas3 // B_{2,3} // B_{1,3} // B_{1,2}
{v_1, v_1 - t v_1 + t v_2, v_1 - t v_1 + t v_2 - t^2 v_2 + t^2 v_3}
```
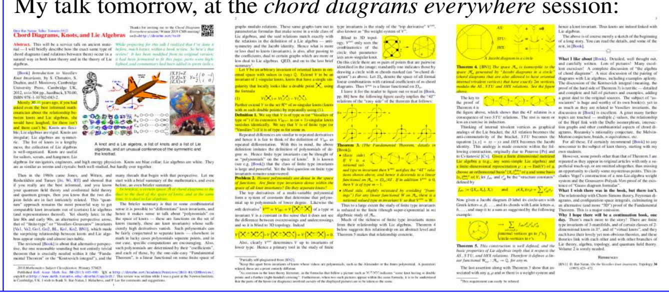

S_n acts on R^n by permuting the v_i so the Burau representation extends to vB_n and restricts to B_n. With this, γ_i maps $v_i \mapsto v_{i+1}$, $v_{i+1} \mapsto tv_i + (1-t)v_{i+1}$, and otherwise $v_k \mapsto v_k$.

My talk tomorrow, at the *chord diagrams everywhere* session:



More Dror: ωεβ/talks

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/