$\mathcal{D}\colon \operatorname{Hom}(U^{\otimes\Sigma} \to U^{\otimes S}) \to \mathbb{Q}[\![\eta_\Sigma, \beta_\Sigma, \alpha_\Sigma, \xi_\Sigma, y_S, b_S, a_S, x_S]\!]$. The PBW theorem for CU (always in the *ybax* order), or its quantum analog for QU, say that if $U = CU$ or QU then $U^{\otimes S}$ is isomorphic as a vector space to $\mathbb{Q}[y_i, b_i, a_i, x_i]_{i\in S}[\![\hbar]\!]$; so it is enough to understand $\operatorname{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$ for finite sets A and B.

Claim. $F \in \operatorname{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B]) \xrightarrow[\mathcal{D}]{\sim} \mathbb{Q}[z_B][\![\zeta_A]\!] \ni \mathcal{F}$ via

$$\mathcal{D}(F) := \sum_{n\in\mathbb{N}^A} \frac{\zeta_A^n}{n!} F(z_A^n) = F\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{F},$$

$$\mathcal{D}^{-1}(\mathcal{F})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{F}\right)_{\zeta_a=0} \quad \text{for } p \in \mathbb{Q}[z_A].$$

Claim. Assuming convergence, if $F \in \operatorname{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$, $G \in \operatorname{Hom}(\mathbb{Q}[z_B] \to \mathbb{Q}[z_C])$, $\mathcal{F} = \mathcal{D}(F)$, and $\mathcal{G} = \mathcal{D}(G)$, then

$$\mathcal{D}(F/\!/G) = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

And so the title of the talk finally makes sense!

Example. $\mathcal{D}(id\colon U \to U) = \mathbb{e}^{\eta y+\beta b+\alpha a+\xi x}$.

Example. Let $c\Delta^i_{jk}\colon CU^{\otimes\{i\}} \to CU^{\otimes\{j,k\}}$ be the standard coproduct, given by $c\Delta^i_{jk}(y_i, b_i, a_i, x_i) = (y_j+y_k, b_j+b_k, a_j+a_k, x_j+x_k)$. Then

$$\mathcal{D}(c\Delta^i_{jk}) = c\Delta^i_{jk}(\mathbb{e}^{\eta_i y_i+\beta_i b_i+\alpha_i a_i+\xi_i x_i})$$
$$= \mathbb{e}^{\eta_i(y_j+y_k)+\beta_i(b_j+b_k)+\alpha_i(a_j+a_k)+\xi_i(x_j+x_k)}.$$

Example. The standard commutative product m^{ij}_k of polynomials is given by $z_i, z_j \to z_k$. Hence $\mathcal{D}(m^{ij}_k) = m^{ij}_k(\mathbb{e}^{\zeta_i z_i+\zeta_j z_j}) = \mathbb{e}^{(\zeta_i+\zeta_j)z_k}$.

$$\begin{array}{ccc} \mathbb{Q}[z]_i \otimes \mathbb{Q}[z]_j & \xrightarrow{m^{ij}_k} & \mathbb{Q}[z]_k \\ \| & & \| \\ \mathbb{Q}[z_i, z_j] & \xrightarrow{m^{ij}_k} & \mathbb{Q}[z_k] \end{array}$$

A real DoPeGDO Example. Let $cm^{ij}_k\colon CU_i \otimes CU_j \to CU_k$ be "classical multiplication" for sl^ϵ_{2+}, and let $\mathbb{O}_i\colon \mathbb{Q}[y_i, b_i, a_i, x_i] \to CU_i$ be the PBW ordering map.

$$\begin{array}{ccc} CU_i \otimes CU_j & \xrightarrow{cm^{ij}_k} & CU_k \\ \uparrow \mathbb{O}_{i,j} & & \uparrow \mathbb{O}_k \\ \mathbb{Q}[y_i, b_i, a_i, x_i, y_j, b_j, a_j, x_j] & & \mathbb{Q}[y_k, b_k, a_k, x_k] \end{array}$$

Claim. Let (all brawn and no brains)

$$\Lambda = \left(\eta_i + \frac{e^{-\alpha_i-\epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right)y_k + \left(\beta_i+\beta_j+\frac{\log\left(1+\epsilon\eta_j\xi_i\right)}{\epsilon}\right)b_k +$$
$$\left(\alpha_i+\alpha_j+\log\left(1+\epsilon\eta_j\xi_i\right)\right)a_k + \left(\frac{e^{-\alpha_j-\epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i}+\xi_j\right)x_k$$

Then $\mathbb{e}^{\eta_i y_i+\beta_i b_i+\alpha_i a_i+\xi_i x_i+\eta_j y_j+\beta_j b_j+\alpha_j a_j+\xi_j x_j}/\!/\mathbb{O}_{i,j}/\!/cm^{ij}_k = \mathbb{e}^\Lambda/\!/\mathbb{O}_k$, and hence $\mathcal{D}(cm^{ij}_k) = \mathbb{e}^\Lambda$ and cm^{ij}_k is DoPeGDO.

Proof. We compute in a faithful 2D representation $z \mapsto \hat{z}$ of CU: (ωεβ/cm)

```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
{ŷ = (0 0; ϵ 0), b̂ = (0 0; 0 -ϵ), â = (1 0; 0 0), x̂ = (0 1; 0 0)};
HL /@ {â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -ϵ ŷ,
  b̂.x̂ - x̂.b̂ == ϵ x̂, x̂.ŷ - ŷ.x̂ == b̂ + ϵ â}

{True, True, True, True, True}
HL@Simplify@With[{𝔼 = MatrixExp},
  𝔼[ηi ŷ].𝔼[βi b̂].𝔼[αi â].𝔼[ξi x̂].𝔼[ηj ŷ].𝔼[βj b̂].
   𝔼[αj â].𝔼[ξj x̂] == 𝔼[ŷ ∂yk Λ].𝔼[b̂ ∂bk Λ].𝔼[â ∂ak Λ].
   𝔼[x̂ ∂xk Λ]]

True
Series[Λ, {ϵ, 0, 1}]
(ak (αi + αj) + yk (ηi + e^(-αi) ηj) +
   bk (βi + βj + ηj ξi) + xk (e^(-αj) ξi + ξj)) +
 (ak ηj ξi - 1/2 bk ηj^2 ξi^2 - e^(-αi) yk ηj (βi + ηj ξi) -
   e^(-αj) xk ξi (βj + ηj ξi)) ϵ + O[ϵ]^2
```

(Shame, but this technique fails for QU).

Claim. In QU, R is DoPeGDO.

Proof. Recall that with $q = \mathbb{e}^{\hbar\epsilon}$,

$$R = \sum \hbar^{j+k}y^k b^j \otimes a^j x^k / j![k]_q! = \mathbb{O}\left(\mathbb{e}^{\hbar b_1 a_2}\mathbb{e}_q^{\hbar y_1 x_2}\right).$$

Now expand $\mathbb{e}_q^{\hbar y_1 x_2}$ in powers of ϵ using:

Faddeev's Formula (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n-1}{q-1}$, with $[n]_q! := [1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x := \sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we have

$$\log \mathbb{e}_q^x = \sum_{k\geq 1}\frac{(1-q)^k x^k}{k(1-q^k)} = x + \frac{(1-q)^2x^2}{2(1-q^2)} + \dots.$$

Proof. We have that $\mathbb{e}_q^x = \frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx} = (1+(1-q)x)\mathbb{e}_q^x$, and

$$\log \mathbb{e}_q^{qx} = \log(1+(1-q)x) + \log \mathbb{e}_q^x.$$

Writing $\log \mathbb{e}_q^x = \sum_{k\geq 1} a_k x^k$ and comparing powers of x, we get $q^k a_k = -(1-q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. □

Compositions (2). Recall that with all indices i running in some set B,

$$\mathcal{F}/\!/\mathcal{G} = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0} \overset{(1)}{=} \mathbb{e}^{\sum\partial_{z_i}\partial_{\zeta_i}}(\mathcal{F}\mathcal{G})\Big|_{z_i=\zeta_i=0},$$

(1) Strictly speaking, true only when $B \cap (A \cup C) = \emptyset$.

so in general we wish to understand

$$[F\colon \mathcal{E}]_B := \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad\text{and}\quad \langle F\colon \mathcal{E}\rangle_B := [F\colon \mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F\colon \mathcal{E}\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}\colon \mathcal{E}\right\rangle_B.$$

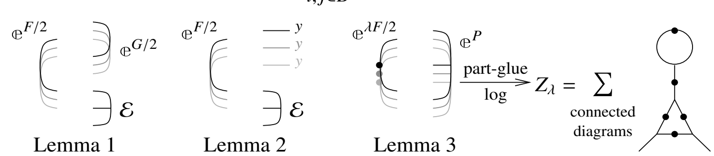
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F\colon \mathcal{E}\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F\colon\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B$.

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F\colon \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad\text{and}\quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Complexity to ϵ^k, for an n-xing width w knot (by [LT], $w \in O(\sqrt{n})$), is $O(n^2w^{2k+2}\log n) = O(n^{k+3}\log n)$ integer operations.

Video and more: http://www.math.toronto.edu/~drorbn/Talks/CRM-1907, http://www.math.toronto.edu/~drorbn/Talks/UCLA-191101.