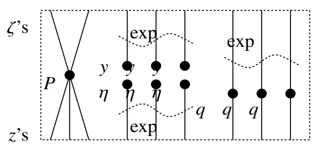
A Partial To Do List.

- Understand tr and links.
- Implement Φ, J. Determine the appropriate wt-0 ground ring.
- Implement the "dequantizators".
- Understand denominators and get rid of them.
- Implement zipping at the log-level.
- Clean the program and make it efficient.
- Run it for all small knots and links, at $k = 3, 4$.
- Understand the centre and figure out how to read the output.
- Is the "+" really necessary in sl^{ϵ}_{2+}? Why?
- Extend to sl_3 and beyond.
- Describe a genus bound and a Seifert formula.
- Obtain "Gauss-Gassner formulas" (ωεβ/NCSU).
- Relate with the representation theory dogma, with Melvin-Morton-Rozansky and with Rozansky-Overbay.
- Understand the braid group representations that arise.
- Relate with finite-type (Vassiliev) invariants.
- Find a topological interpretation/foundation. The Garoufalidis - Rozansky "loop expansion" [GR]?
- Figure out the action of the Cartan automorphism.
- Understand "the subspace of classical knots / tangles".
- Disprove the ribbon-slice conjecture!
- Figure out the action of the Weyl group.
- Use to study "Ševera quantization".
- Do everything at the "arrow diagram" level of finite-type invariants of (rotational) virtual tangles.
- Find "internal" proofs of consistency.
- What else can you do with the "solvable approximations"?
- And with the "Gaussian compositions" technology?

Warning. Some implementation details match earlier versions of the theory.

The Zipping Theorem. If P has a finite ζ-degree and $\tilde{q}$ is the inverse matrix of $1-q$: $(\delta^i_j - q^i_j)\tilde{q}^j_k = \delta^i_k$, then



$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i + y_j\zeta^j + q^i_j z_i \zeta^j}\right\rangle$$
$$= |\tilde{q}|\mathbb{e}^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k), \zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

The "Speedy" Engine

ωεβ/engine

Internal Utilities

Canonical Form:

```
CCF[ℰ_] :=
  PP_CCF@ExpandDenominator@
    ExpandNumerator@PP_Together@Together[PP_Exp[
        Expand[ℰ] //. 𝕖^x_ 𝕖^y_ :> 𝕖^(x+y) /. 𝕖^x_ :> 𝕖^CCF[x]]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := PP_CF@Module[
    {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ)_, ∞] ⋃
       {y, b, t, a, x, η, β, τ, α, ξ}},
    Total[CoefficientRules[Expand[ℰ], vs] /.
      (ps_ → c_) :> CCF[c] (Times @@ vs^ps)]
   ];
CF[ℰ_𝔼] := CF /@ ℰ;
CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker δ:

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $\mathbb{e}^{L+Q}P$:

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] × 𝔼[L2_, Q2_, P2_] :=
  𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z};
(u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

```
U2l = {B_i_^p_. :> 𝕖^(-p ℏ γ b_i), B^p_. :> 𝕖^(-p ℏ γ b), T_i_^p_. :> 𝕖^(p ℏ t_i),
   T^p_. :> 𝕖^(p ℏ t), 𝒜_i_^p_. :> 𝕖^(p γ α_i), 𝒜^p_. :> 𝕖^(p γ α)};
l2U = {𝕖^(c_. b_i_ + d_.) :> B_i^(-c/(ℏ γ)) 𝕖^d, 𝕖^(c_. b + d_.) :> B^(-c/(ℏ γ)) 𝕖^d,
   𝕖^(c_. t_i_ + d_.) :> T_i^(c/ℏ) 𝕖^d, 𝕖^(c_. t + d_.) :> T^(c/ℏ) 𝕖^d,
   𝕖^(c_. α_i_ + d_.) :> 𝒜_i^(c/γ) 𝕖^d, 𝕖^(c_. α + d_.) :> 𝒜^(c/γ) 𝕖^d,
   𝕖^ℰ_ :> 𝕖^Expand@ℰ};
```

Derivatives in the presence of exponentiated variables:

```
D_b[f_] := ∂_b f - ℏ γ B ∂_B f; D_{b_i_}[f_] := ∂_{b_i} f - ℏ γ B_i ∂_{B_i} f;
D_t[f_] := ∂_t f + ℏ T ∂_T f; D_{t_i_}[f_] := ∂_{t_i} f + ℏ T_i ∂_{T_i} f;
D_α[f_] := ∂_α f + γ 𝒜 ∂_𝒜 f; D_{α_i_}[f_] := ∂_{α_i} f + γ 𝒜_i ∂_{𝒜_i} f;
D_v_[f_] := ∂_v f; D_{v_,0}[f_] := f; D_{}[f_] := f;
D_{v_,n_Integer}[f_] := D_v[D_{v,n-1}[f]];
D_{l_List,ls___}[f_] := D_{ls}[D_l[f]];
```

Finite Zips:

```
collect[sd_SeriesData, ζ_] :=
  MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := PP_Collect@Collect[ℰ, ζ];
Zip_{}[P_] := P;
Zip_ζs_[Ps_List] := Zip_ζs /@ Ps;
Zip_{ζ_,ζs___}[P_] := PP_Zip[
  (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :> (D_{ζ*,d}[f])) /.
    ζ* → 0 /. ((ζ* /. {b → B, t → T, α → 𝒜}) → 1)]
```

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = \mathbb{e}^{L+Q}P(\epsilon)$. Such zips regard the L variables as scalars.