$$\mathrm{dm}_{i,j\to k}\to a_k\,(\alpha_i+\alpha_j)+b_k\,(\beta_i+\beta_j)+y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}+\eta_j\,\xi_i-B_k\,\eta_j\,\xi_i+\frac{1}{4\,\mathcal{A}_i\,\mathcal{A}_j}\in\Big(2\,y_k\,\eta_j\,(2\,x_k\,\xi_i+\mathcal{A}_j\,(-2\,\beta_i+(1-3\,B_k)\,\eta_j\,\xi_i))+\mathcal{A}_i\,\xi_i\,\big(x_k\,(-4\,\beta_j+2\,(1-3\,B_k)\,\eta_j\,\xi_i)+\mathcal{A}_j\,\eta_j\,\big(4\,a_k\,B_k+\big(1-4\,B_k+3\,B_k^2\big)\,\eta_j\,\xi_i\big)\big)\Big)+x_k\,\xi_j$$

$$\mathrm{d}\triangle_{i\to j,k}\to a_j\,\alpha_i+a_k\,\alpha_i+b_j\,\beta_i+b_k\,\beta_i+y_j\,\eta_i+B_j\,y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i+\frac{1}{2}\in\big(B_j\,y_j\,y_k\,\eta_i^2+x_k\,\xi_i\,(-2\,a_j+x_j\,\xi_i)\big)$$

$$\mathrm{dS}_i\to -a_i\,\alpha_i-b_i\,\beta_i-\frac{\mathcal{A}_i\,(y_i\,\eta_i+(-\eta_i+B_i\,(x_i+\eta_i))\,\xi_i)}{B_i}-\frac{1}{4\,B_i^2}\in\mathcal{A}_i\,\Big(\mathcal{A}_i\,\eta_i^2\,\big(2\,y_i^2-6\,y_i\,\xi_i+3\,\xi_i^2\big)+B_i^2\,\xi_i\,\big(4\,a_i\,x_i+2\,x_i^2\,\mathcal{A}_i\,\xi_i+2\,x_i\,(2\,\beta_i+\mathcal{A}_i\,\eta_i\,\xi_i)+\eta_i\,(-4+4\,\beta_i+\mathcal{A}_i\,\eta_i\,\xi_i)\big)+2\,B_i\,\eta_i\,(y_i\,(-2+2\,\beta_i+2\,x_i\,\mathcal{A}_i\,\xi_i+\mathcal{A}_i\,\eta_i\,\xi_i)-\xi_i\,(-2+2\,a_i+2\,\beta_i+3\,x_i\,\mathcal{A}_i\,\xi_i+2\,\mathcal{A}_i\,\eta_i\,\xi_i))\Big)$$

$$R_{i,j}\to a_j\,b_i+x_j\,y_i-\frac{1}{4}\in x_j^2\,y_i^2$$

$$P_{i,j}\to \alpha_j\,\beta_i+\eta_i\,\xi_j+\frac{1}{4}\in\eta_i^2\,\xi_j^2$$

A Quantum Algebra Example. ωεβ/qa

Proto-Proposition†0 (with Jesse Frohlich and Roland van der Veen, near [Ma, Proposition 1.7.3]). Let H be a finite dimensional Hopf algebra and let $U = H^{*cop}\otimes H$ be its Drinfel'd double, with R-matrix $R\in H^*\otimes H\subset U\otimes U$. Write $R^{\dagger 1}=\sum\rho_a\otimes r_a$, and let $\langle\cdot\mid\cdot\rangle\colon H^*\otimes H\to\mathbb{F}$ be the duality pairing. Then the functional $\int\in U^*$ defined by

$$\int\phi\otimes x := \sum\langle\phi\rho_a^{\dagger 2}\mid x r_a^{\dagger 3}\rangle$$

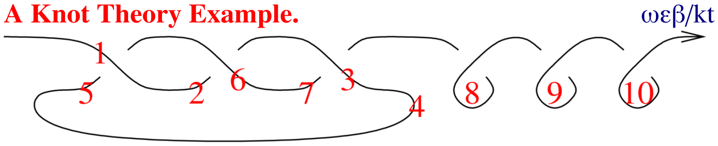
is a right†4 integral in U^*. (Meaning $\Delta^i_{jk}/\!\!/\int_j = \int_i/\!\!/\epsilon_k$ in $\mathrm{Hom}(U^{\otimes\{i\}}\to U^{\otimes\{k\}})$).

†0 A "proto-proposition" is something that will become a proposition once you figure out the correct statement. †1 Or did we want it to be $R/\!\!/S_1^2$? Or $R/\!\!/S_2^2$? †2 Or is it $\rho_a\phi$? †3 Or is it $r_a x$? †4 Or maybe "left"?

```
inp = E_{}→{1}[3 a_1 b_1, 5 x_1 y_1, 1] // dm_{i,1→i};
Table[
  HL@TrueQ[
    (inp // (sΥ_{i→1,1,2,2} RR) // BM // AM // P_{1,2} ) dϵ_j ≡
     (inp // ΔΔ // (sΥ_{i→1,1,2,2} RR) // BM // AM // P_{1,2})],
  {ΔΔ, {dΔ_{i→i,j}, dΔ_{i→j,i}}}, {AM, {dm_{2,4→2}, dm_{4,2→2}}},
  {BM, {dm_{1,3→1}, dm_{3,1→1}}},
  {RR, {R_{3,4}, R_{3,4} // dS_3 // dS_3, R_{3,4} // dS_4 // dS_4}}
 ] // MatrixForm
```

False	False	False	False	False	True
False	False	False	False	False	False
False	False	False	False	False	False
False	False	True	False	False	False

A Knot Theory Example. ωεβ/kt




```
$k = 2;
Simplify[
  R_{1,5} R_{6,2} R_{3,7} C̄_4 K̄ink_8 K̄ink_9 K̄ink_10 // dm_{1,2→1} // dm_{1,3→1} //
        dm_{1,4→1} // dm_{1,5→1} // dm_{1,6→1} // dm_{1,7→1} // dm_{1,8→1} //
    dm_{1,9→1} // dm_{1,10→1}] /. v_1 :→ v
```

$$\mathbb{E}_{\{\}\to\{1\}}\Big[0,\,0,\,\frac{B}{1-B+B^2}+\frac{B\,\big(-B+2\,B^2+2\,B^4+a\,\big(-1+B-B^3+B^4\big)-2\,x\,y-B^3\,(3+2\,x\,y)\big)\in}{\big(1-B+B^2\big)^3}+\frac{1}{2\,\big(1-B+B^2\big)^5}B\,\Big(4\,B^8+a^2\,\big(1-B+B^2\big)^2\,\big(1+B-6\,B^2+B^3+B^4\big)+6\,B^5\,x^2\,y^2+2\,x\,y\,(-2+3\,x\,y)-B^7\,(11+4\,x\,y)-2\,B^2\,\big(1+6\,x^2\,y^2\big)-2\,B^4\,\big(1-2\,x\,y+6\,x^2\,y^2\big)+B\,\big(1+8\,x\,y+6\,x^2\,y^2\big)+B^6\,\big(6+8\,x\,y+6\,x^2\,y^2\big)+B^3\,\big(4+4\,x\,y+30\,x^2\,y^2\big)+2\,a\,\big(1-B+B^2\big)\,\big(2\,B^6+2\,x\,y+8\,B^3\,(1+x\,y)-5\,B^2\,(1+2\,x\,y)-2\,B^5\,(1+2\,x\,y)-B^4\,(7+2\,x\,y)+B\,(2+4\,x\,y)\big)\Big)\in^2+O[\in]^3\Big]$$

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