Computation without Representation

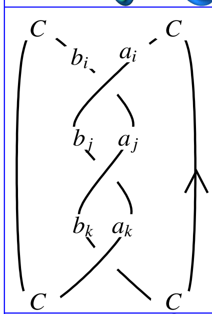
Thanks for inviting me to Ohio!
ωεβ:=http://drorbn.net/o19/

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at [BV] and at ωεβ/talks.

Abstract. A major part of "quantum topology" is the definition and computation of various knot invariants by carrying out computations in quantum groups. Traditionally these computations are carried out "in a representation", but this is very slow: one has to use tensor powers of these representations, and the dimensions of powers grow exponentially fast.
In my talk, I will describe a direct method for carrying out such computations without having to choose a representation and explain why in many ways the results are better and faster. The two key points we use are a technique for composing infinite-order "perturbed Gaussian" differential operators, and the little-known fact that every semi-simple Lie algebra can be approximated by solvable Lie algebras, where computations are easier.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know. (experimental analysis @ωεβ/kiw)

Knotted Candies ωεβ/kc



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R=\sum a_i\otimes b_i\in U\otimes U \quad\text{and}\quad C\in U,$$

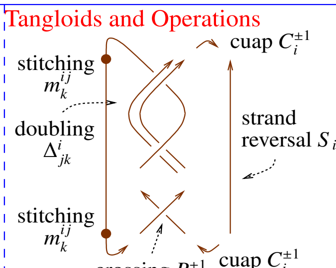
form
$$Z=\sum_{i,j,k} Ca_ib_ja_kC^2b_ia_jb_kC.$$

Problem. Extract information from Z.
The Dogma. Use representation theory. In principle finite, but *slow*.

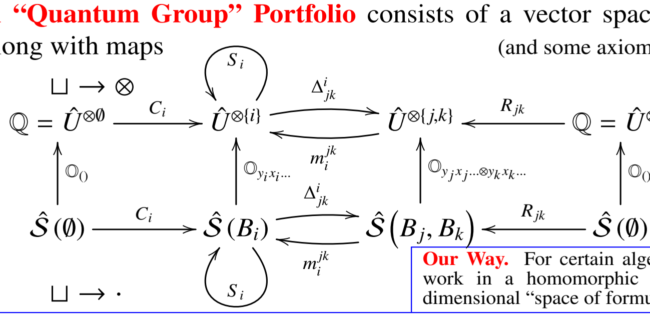
A Knot Theory Portfolio.

- Has operations $\sqcup$, m^{ij}_k, Δ^i_{jk}, S_i.
- All tangloids are generated by $R^{\pm1}$ and $C^{\pm1}$ (so "easy" to produce invariants).
- Makes some knot properties ("genus", "ribbon") become "definable".

Tangloids and Operations

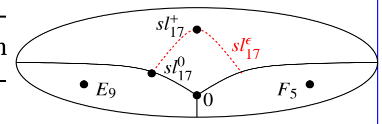


A "Quantum Group" Portfolio consists of a vector space U along with maps (and some axioms...)

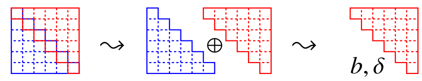


Our Way. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes2}\otimes V$ is on the right. We care about $sl^k_{17}:=sl^\epsilon_{17}/(\epsilon^{k+1}=0)$.



Solvable Approximation. In gl_n, half is enough! Indeed $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(\triangledown,b,\delta)$:



Now define $gl^\epsilon_n:=\mathcal{D}(\triangledown,b,\epsilon\delta)$. Schematically, this is $[\triangledown,\triangledown]=\triangledown$, $[\triangle,\triangle]=\epsilon\triangle$, and $[\triangledown,\triangle]=\triangle+\epsilon\triangledown$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1}=0$ always yields a solvable Lie algebra.

CU and QU. Starting from sl_2, get $CU_\epsilon=\langle y,a,x,t\rangle/([t,-]=0,\ [a,y]=-y,\ [a,x]=x,\ [x,y]=2\epsilon a-t)$. Quantize using standard tools (I'm sorry) and get $QU_\epsilon=\langle y,a,x,t\rangle/([t,-]=0,\ [a,y]=-y,\ [a,x]=x,\ xy-\mathbb{e}^{\hbar\epsilon}yx=(1-T\mathbb{e}^{-2\hbar\epsilon a})/\hbar)$.

PBW Bases. The U's we care about always have "Poincaré-Birkhoff-Witt" bases; there is some finite set $B=\{y,x,\ldots\}$ of "generators" and isomorphisms $\mathbb{O}_{y,x,\ldots}\colon\hat{\mathcal{S}}(B)\to U$ defined by "ordering monomials" to some fixed $y,x,\ldots$ order. The quantum group portfolio now becomes a "symmetric algebra" portfolio, or a "power series" portfolio.

Operations are Objects.

★ $B^*:=\{z^*_i=\zeta_i\colon z_i\in B\}$,

$$\langle z^m_i,\zeta^n_i\rangle=\delta_{mn}n!,$$

$$\left\langle\prod z^{m_i}_i,\prod\zeta^{n_i}_i\right\rangle=\prod\delta_{m_in_i}n_i!,$$

in general, for $f\in\mathcal{S}(z_i)$ and $g\in\mathcal{S}(\zeta_i)$,

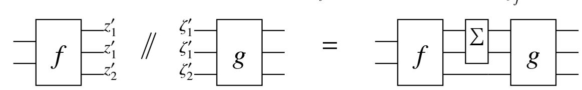
$$\langle f,g\rangle=f(\partial_{\zeta_i})g\big|_{\zeta_i=0}=g(\partial_{z_i})f\big|_{z_i=0}.$$

$f\in\mathrm{Hom}_{\mathbb{Q}}(S(B)\to S(B'))$ $\;\|\;$ $S(B)^*\otimes S(B')$ $\;\|$★$\;$ $S(B^*)\otimes S(B')$ $\;\|\;$ $S(B^*\sqcup B')$ $\;\|\;$ $\tilde{f}\in\mathbb{Q}[\zeta_i,z'_i]$

The Composition Law. If

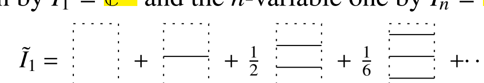
$$\mathcal{S}(B)\xrightarrow[\tilde f\in\mathbb{Q}[\![\zeta_i,z'_j]\!]]{f}\mathcal{S}(B')\xrightarrow[\tilde g\in\mathbb{Q}[\![\zeta'_j,z''_k]\!]]{g}\mathcal{S}(B'')$$

then
$$\widetilde{(f/\!\!/g)}=\widetilde{(g\circ f)}=\left(\tilde g\big|_{\zeta'_j\to\partial_{z'_j}}\tilde f\right)_{z'_j=0}=\left(\tilde f\big|_{z'_j\to\partial_{\zeta'_j}}\tilde g\right)_{\zeta'_j=0}:$$

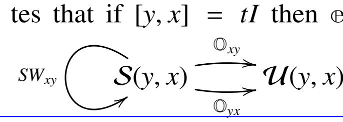


Examples

1. The 1-variable identity map $I\colon\mathcal{S}(z)\to\mathcal{S}(z)$ is given by $\tilde I_1=\mathbb{e}^{z\zeta}$ and the n-variable one by $\tilde I_n=\mathbb{e}^{z_1\zeta_1+\cdots+z_n\zeta_n}$:



2. The "archetypal multiplication map $m^{ij}_k\colon\mathcal{S}(z_i,z_j)\to\mathcal{S}(z_k)$" has $\tilde m=\mathbb{e}^{z_k(\zeta_i+\zeta_j)}$.
3. The "archetypal coproduct $\Delta^i_{jk}\colon\mathcal{S}(z_i)\to\mathcal{S}(z_j,z_k)$", given by $z_i\to z_j+z_k$ or $\Delta z=z\otimes1+1\otimes z$, has $\tilde\Delta=\mathbb{e}^{(z_j+z_k)\zeta_i}$.
4. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1x_2}\in\mathcal{U}_q\otimes\mathcal{U}_q$. The "baby R-matrix" is $\tilde R=\mathbb{e}^{\hbar yx}\in\mathcal{S}(y,x)$.
5. The "Weyl form of the canonical commutation relations" states that if $[y,x]=tI$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y}=\mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. So with



we have $\widetilde{SW}_{xy}=\mathbb{e}^{\eta y+\xi x-\eta\xi t}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ohio-1901