Solvable Approximations of the Quantum sl_2 Portfolio

With Roland van der Veen

ωεβ:=http://drorbn.net/mm18/

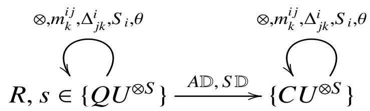
W.I.P. Warning / Brute Warning

Our Main Theorem (loosely stated). Everything that matters in the quantum sl_2 portfolio can be continuously expressed in terms of docile perturbed Gaussians using solvable approximations. ○

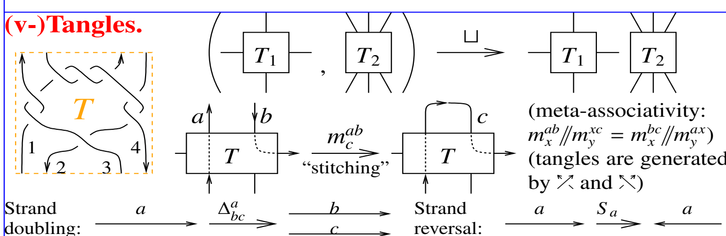
Our Main Points.

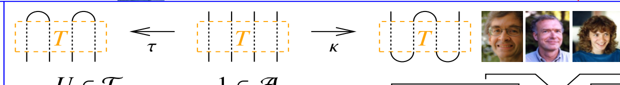
- What's the "quantum sl_2 portfolio"?
- What in it "matters" and why? (the most important question)
- What's "solvable approximation"? What's "continuously"?
- What are "docile perturbed Gaussians"?
- Why do they matter? (2^{nd} most important)
- How proven? (docile)
- How implemented? (sacred; the work of unsung heroes)
- Some context and background.
- What's next?

The quantum sl_2 Portfolio includes a classical universal enveloping algebra CU, its quantization QU, their tensor powers $CU^{\otimes S}$ and $QU^{\otimes S}$ with the "tensor operations" ⊗, their products m^{ij}_k, coproducts Δ^i_{jk} and antipodes S_i, their Cartan automophisms $C\theta\colon CU\to CU$ and $Q\theta\colon QU\to QU$, the "dequantizators" $A\mathbb{D}\colon QU\to CU$ and $S\mathbb{D}\colon QU\to CU$, and most importantly, the R-matrix R and the Drinfel'd element s. All this in any PBW basis, and change of basis maps are included.



(v-)Tangles.



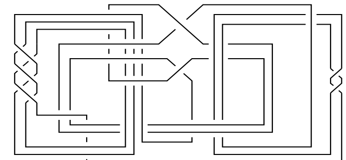


$U\in\mathcal{T}_n$ $\quad 1\in\mathcal{A}_n$

$\mathcal{T}_{2n}\xrightarrow{z}\mathcal{A}_{2n}$ with $\mathcal{R}:=\kappa(\tau_a^{-1}(1))$

ribbon $K\in\mathcal{T}_1$ $\quad z(K)\in\mathcal{R}\subseteq\mathcal{A}_1$

Faster is better, leaner is meaner!



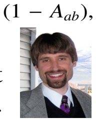
The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has

$$\Gamma(T)\in R_S\times M_{S\times S}(R_S)=\left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}\text{ with } R_S:=\mathbb{Z}(\{t_a\colon a\in S\}):$$

$$\left({}_a\!\nearrow\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\nearrow_a\right)\to\begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-t_a^{\pm1}\\ b & 0 & t_a^{\pm1}\end{array}\qquad T_1\sqcup T_2\to\begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array}\xrightarrow[t_a,t_b\to t_c]{m^{ab}_c}\left(\begin{array}{c|cc}(1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

(Roland: "add to A the product of column b and row a, divide by $(1-A_{ab})$, delete column b and row a".)



For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

$$\begin{array}{c|cc}\omega & a & S\\ \hline a & \alpha & \theta\\ S & \phi & \Xi\end{array}\xrightarrow[\substack{\mu:=T_a-1\\ \nu:=\alpha-\sigma_a\\ T_a\mapsto T_bT_c}]{q\Delta^a_{bc}}\left(\begin{array}{c|ccc}\omega & b & c & S\\ \hline b & (\sigma_a-\alpha T_a-\nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu\\ c & (T_c-1)\nu/\mu & (\alpha-\sigma_aT_a-\nu T_c)/\mu & (T_c-1)\theta/\mu\\ S & \phi & \phi & \Xi\end{array}\right)$$

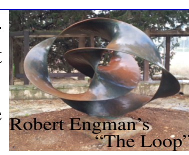
$$dS^a\Big\downarrow T_a\to T_a^{-1}$$

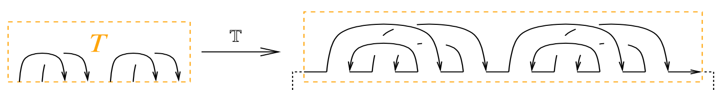
$$\left(\begin{array}{c|cc}\alpha\omega/\sigma_a & a & S\\ \hline a & 1/\alpha & \theta/\alpha\\ S & -\phi/\alpha & (\alpha\Xi-\phi\theta)/\alpha\end{array}\right)$$

Where σ assigns to every $a\in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left({}_a\!\nearrow\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\nearrow_a\right)=(a\to1,\ b\to t_a^{\pm1})$, $\sigma(T_1\sqcup T_2)=\sigma(T_1)\sqcup\sigma(T_2)$, and $\sigma/\!\!/m^{ab}_c=(\sigma\setminus\{a,b\})\cup(c\to\sigma_a\sigma_b)|_{t_a,t_b\to t_c}$.

Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.
Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via





Implementation key idea: ωεβ/AlexDemo

$(\omega, A=(\alpha_{ab}))\leftrightarrow$
$(\omega,\lambda=\sum\alpha_{ab}t_ah_b)$

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Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_,b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂_{t_a,h_a}λ ∂_{t_a,h_b}λ ∂_{t_a}λ; ∂_{t_b,h_a}λ ∂_{t_b,h_b}λ ∂_{t_b}λ; ∂_{h_a}λ ∂_{h_b}λ λ) /. (t | h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ / μ   ϵ + δ θ / μ; ϕ + α ψ / μ   Ξ + ψ θ / μ).{h_c, 1}]
    /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.(1  1 - T_a; 0  T_a).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1 / T_a;

ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_{a_} ↦ a, ∞];
   M = Outer[Factor[∂_{h_{#1} t_{#2}} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
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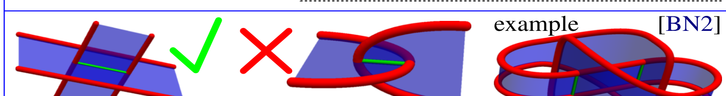

A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3=\partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,
Conjecture. Some slice knots are not ribbon.
Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t)=f(t)f(1/t)$. (also for slice)

Meta-Associativity **Runs.**

```
ξ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];

(ξ // m12→1 // m13→1) == (ξ // m23→2 // m12→1)

True
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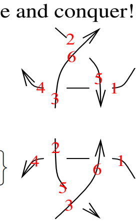
R3 ... divide and conquer!

```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix},\ \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$

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z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```
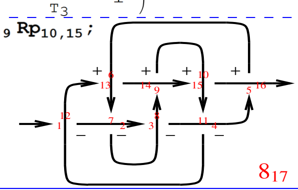
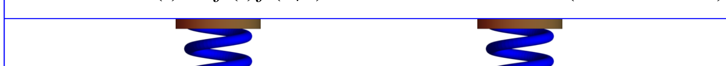
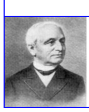

$$\begin{pmatrix}11-\frac{1}{T_1^3}+\frac{4}{T_1^2}-\frac{8}{T_1}-8T_1+4T_1^2-T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$







"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/