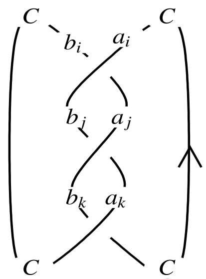


The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

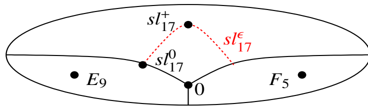
$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

Problem. Extract information from Z.
The Dogma. Use representation theory. In principle finite, but *slow*.

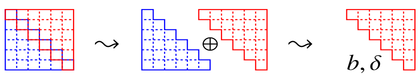
The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2}\otimes V$ is on the right. We care about $sl^k_{17} := sl^\epsilon_{17}/(\epsilon^{k+1}=0)$.



Solvable Approximation. A quantized universal enveloping algebra (aka "quantum group") is an ∞-dimensional inverse limit.

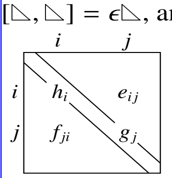
untrimmed	$\longrightarrow$	halfway-trimmed	$\longrightarrow$	almost fully-trimmed
$\updownarrow$		$\updownarrow$		$\updownarrow$
too hard		"solvable approximation" (a parameter is hidden)		"finite-type"

Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b\colon ◹ \otimes ◹ \to ◹$
$b(◺) \rightsquigarrow \delta\colon ◹ \to ◹ \otimes ◹$

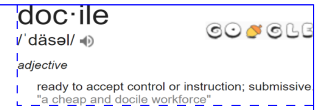
Now define $gl^\epsilon_n := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon ◺$, and $[◹, ◺] = ◺ + \epsilon ◹$. In detail, it is



$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon\delta_{jk} f_{il} - \epsilon\delta_{li} f_{kj}$$
$$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{j<k} e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l} f_{il})$$
$$-\delta_{li}(\epsilon\delta_{k<j} e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j} f_{kj})$$
$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik}) e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) e_{jk}$$
$$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik}) f_{jk} \qquad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) f_{jk}$$

Solvable Approximation (2). At $\epsilon = 1$ and modulo $h = g$, the above is just gl_n. By rescaling at $\epsilon \neq 0$, gl^ϵ_n is independent of ϵ. We let gl^k_n be gl^ϵ_n regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the "$k$-smidgen solvable approximation" of gl_n!
Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2, \mathfrak{g}_2]$, $\dots$, $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.
Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

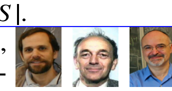
Definition. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form



$$\mathbb{e}^{q^{ij} z_i z_j} P = \mathbb{e}^{q^{ij} z_i z_j} \left(\sum_{k\geq 0} \epsilon^k P_k \right),$$

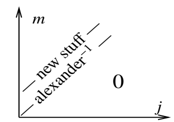
where all coefficients are in R and where P is a "docile series": $\deg P_k \leq 4k$.
Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing


Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2} - q^{-1/2}) J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K) d^j \hbar^m,$$

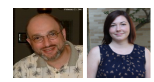
"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot \omega(K)(e^\hbar) = 1$.



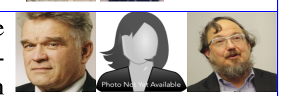
"Above diagonal" we have **Rozansky's Theorem** [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})\omega(K)(q^d)} \left(1 + \sum_{k=1}^\infty \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)} \right).$$

Prior art. Some amazing computations by Rozansky and Overbay in [Ro2, Ro3] and in [Ov].


Rozansky, Overbay

Faddeev's Formula (In as much as we can tell, first appeared w/o proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n - 1}{q-1}$, with $[n]_q! := [1]_q[2]_q \cdots [n]_q$ and with $\mathbb{e}^x_q := \sum_{n\geq 0} \frac{x^n}{[n]_q!}$, we have


Faddeev Quesne Zagier

$$\log \mathbb{e}^x_q = \sum_{k\geq 1} \frac{(1-q)^k x^k}{k(1-q^k)} = x + \frac{(1-q)^2 x^2}{2(1-q^2)} + \dots.$$

Proof. We have that $\mathbb{e}^x_q = \frac{\mathbb{e}^{qx}_q - \mathbb{e}^x_q}{qx - x}$ ("the q-derivative of $\mathbb{e}^x_q$ is itself"), and hence $\mathbb{e}^{qx}_q = (1 + (1-q)x)\mathbb{e}^x_q$, and

$$\log \mathbb{e}^{qx}_q = \log(1 + (1-q)x) + \log \mathbb{e}^x_q.$$

Writing $\log \mathbb{e}^x_q = \sum_{k\geq 1} a_k x^k$ and comparing powers of x, we get $q^k a_k = -(1-q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. □

$\mathcal{GDO}$-Categories. Given $\mathfrak{g}$ with basis $B = \{x, y, \dots\}$, consider the following diagram:

$$\begin{tikzcd}
\mathbb{Q} = \hat{\mathcal{U}}_{(q)}\left(\bigoplus_0 \mathfrak{g}\right) \arrow[r, "Z"] & \hat{\mathcal{U}}_{(q)}(\mathfrak{g}) \arrow[r, shift left, "\Delta"] & \hat{\mathcal{U}}_{(q)}\left(\bigoplus_2 \mathfrak{g}\right) \arrow[l, shift left, "m"] \\
\hat{\mathcal{S}}(\emptyset) \arrow[u] \arrow[r, "Z"] & \hat{\mathcal{S}}(B) \arrow[u, shift left, "\mathbb{O}(xy\dots:\,\cdot)"] \arrow[u, shift right, "\mathbb{O}(yx\dots:\,\cdot)"'] \arrow[r, shift left, "\Delta"] \arrow[loop below, "SW_{xy}"] & \hat{\mathcal{S}}(B_1, B_2) \arrow[l, shift left, "m"] \arrow[u, "\mathbb{O}(y_1x_1\dots\otimes y_2x_2\dots:\,\cdot)"']
\end{tikzcd}$$

Hence Z, SW_{xy}, m, Δ, (and likewise S and θ) are morphisms in the *completion* of the monoidal category $\mathcal{F}$ whose objects are finite sets B and whose morphisms are $\mathrm{mor}_{\mathcal{F}}(B, B') := \mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B) \to \mathcal{S}(B')) = \mathcal{S}(B^*, B')$ (by convention, $x^* = \xi$, $y^* = \eta$, etc.). Ergo we need to *consolidate* (at least parts of) said completion.

Aside. "Consolidate" means "give a finite name to an infinite object, and figure out how to sufficiently manipulate such finite names". E.g., solving $f'' = -f$ we encounter and set $\sum \frac{(-1)^k x^{2k}}{(2k)!} \rightsquigarrow \cos x$, $\sum \frac{(-1)^k x^{2k+1}}{(2k+1)!} \rightsquigarrow \sin x$, and then $\cos^2 x + \sin^2 x = 1$ and $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

The Composition Law. If

$$\mathcal{S}(B_0) \xrightarrow[\,^t f \in \mathbb{Q}[\![\zeta_{0i}, z_{1j}]\!]]{f} \mathcal{S}(B_1) \xrightarrow[\,^t g \in \mathbb{Q}[\![\zeta_{1j}, z_{2k}]\!]]{g} \mathcal{S}(B_2)$$

then $^t(f/\!\!/g) = {}^t(g \circ f) = \left(g|_{\zeta_{1j} \to \partial_{z_{1j}}} f \right)_{z_{1j}=0}$.

Examples.

1. The 1-variable identity map $I\colon \mathcal{S}(z) \to \mathcal{S}(z)$ is given by $^tI_1 = \mathbb{e}^{z\zeta}$ and the n-variable one by $^tI_n = \mathbb{e}^{z_1\zeta_1 + \cdots + z_n\zeta_n}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/