$\mathbb{E}\Big[a_3\,\alpha_1 + a_3\,\alpha_2 + t_3\,(\tau_1+\tau_2),$

$y_3\,\eta_1 + e^{-\gamma\alpha_1}\,y_3\,\eta_2 + e^{-\gamma\alpha_2}\,x_3\,\xi_1 + \frac{(1-T_3)\,\eta_2\,\xi_1}{\hbar} + x_3\,\xi_2,$

$1 + \frac{1}{4\hbar}\,\eta_2\,\xi_1\,\big(8\,\hbar\,a_3\,T_3 + 4\,e^{-\gamma\alpha_1-\gamma\alpha_2}\,\gamma\,\hbar^2\,x_3\,y_3 + 2\,e^{-\gamma\alpha_1}\,\gamma\,\hbar\,y_3\,\eta_2 -$
$6\,e^{-\gamma\alpha_1}\,\gamma\,\hbar\,T_3\,y_3\,\eta_2 + 2\,e^{-\gamma\alpha_2}\,\gamma\,\hbar\,x_3\,\xi_1 - 6\,e^{-\gamma\alpha_2}\,\gamma\,\hbar\,T_3\,x_3\,\xi_1 +$
$\gamma\,\eta_2\,\xi_1 - 4\,\gamma\,T_3\,\eta_2\,\xi_1 + 3\,\gamma\,T_3^2\,\eta_2\,\xi_1\big)\,\epsilon + O[\epsilon]^2\Big]$

```
S[U_, kk_] := S[U, kk] = Module[{OE},
   OE = m3,2,1→1[ExpQU1,$k[η, S1[QU[y1]] /. QU → Times]
       ExpQU2,$k[α, S2[QU[a2]] /. QU → Times]
       ExpQU3,$k[ξ, S3[QU[x3]] /. QU → Times]];
   𝔼[-t1 τ1 + OE⟦1⟧, OE⟦2⟧, OE⟦3⟧] /.
    {η → η1, α → α1, ξ → ξ1}];
tSi_ := S[$U, $k] /. {(v : τ | η | α | ξ)1 → vi,
    (v : t | T | y | a | x)1 → vi};
```

```
tS1
```

$\mathbb{E}\Big[-a_1\,\alpha_1 - t_1\,\tau_1,$

$\frac{-e^{\gamma\alpha_1}\,\hbar\,y_1\,\eta_1 - e^{\gamma\alpha_1}\,\hbar\,T_1\,x_1\,\xi_1 + e^{\gamma\alpha_1}\,\eta_1\,\xi_1 - e^{\gamma\alpha_1}\,T_1\,\eta_1\,\xi_1}{\hbar\,T_1},\ 1 +$

$\frac{1}{4\,\hbar\,T_1^2}\,\big(4\,e^{\gamma\alpha_1}\,\gamma\,\hbar^2\,T_1\,y_1\,\eta_1 - 4\,e^{\gamma\alpha_1}\,\hbar^2\,a_1\,T_1\,y_1\,\eta_1 - 2\,e^{2\gamma\alpha_1}\,\gamma\,\hbar^2\,y_1^2\,\eta_1^2 -$
$4\,e^{\gamma\alpha_1}\,\hbar^2\,a_1\,T_1^2\,x_1\,\xi_1 - 4\,e^{\gamma\alpha_1}\,\gamma\,\hbar\,T_1\,\eta_1\,\xi_1 + 8\,e^{\gamma\alpha_1}\,\hbar\,a_1\,T_1\,\eta_1\,\xi_1 +$
$4\,e^{\gamma\alpha_1}\,\gamma\,\hbar\,T_1^2\,\eta_1\,\xi_1 - 4\,e^{2\gamma\alpha_1}\,\gamma\,\hbar^2\,T_1\,x_1\,y_1\,\eta_1\,\xi_1 + 6\,e^{2\gamma\alpha_1}\,\gamma$
$\hbar\,y_1\,\eta_1^2\,\xi_1 - 2\,e^{2\gamma\alpha_1}\,\gamma\,\hbar\,T_1\,y_1\,\eta_1^2\,\xi_1 - 2\,e^{2\gamma\alpha_1}\,\gamma\,\hbar^2\,T_1^2\,x_1^2\,\xi_1^2 +$
$6\,e^{2\gamma\alpha_1}\,\gamma\,\hbar\,T_1\,x_1\,\eta_1\,\xi_1^2 - 2\,e^{2\gamma\alpha_1}\,\gamma\,\hbar\,T_1^2\,x_1\,\eta_1\,\xi_1^2 - 3\,e^{2\gamma\alpha_1}\,\gamma\,\eta_1^2\,\xi_1^2 +$
$4\,e^{2\gamma\alpha_1}\,\gamma\,T_1\,\eta_1^2\,\xi_1^2 - e^{2\gamma\alpha_1}\,\gamma\,T_1^2\,\eta_1^2\,\xi_1^2\big)\,\epsilon + O[\epsilon]^2\Big]$

```
Δ[U_, kk_] := Δ[U, kk] = Module[{OE},
   OE = Block[{$k = kk, $p = kk + 1},
     m1,3,5→1@
      m2,4,6→2@Times[ (* Warning:
         wrong unless $p≥$k+1! *)
        ReplacePart[1 → 0]@
         ExpQU1,$k[η, Δ1→1,2[QU[y1]] /. QU → Times],
        ReplacePart[2 → 0]@
         ExpQU3,$k[α, Δ3→3,4[QU[a3]] /. QU → Times],
        ReplacePart[1 → 0]@
         ExpQU5,$k[ξ, Δ5→5,6[QU[x5]] /. QU → Times]
       ] /. {η → η1, α → α1, ξ → ξ1}];
   𝔼[τ1 (t1 + t2) + α1 (a1 + a2), OE⟦2⟧, OE⟦3⟧]];
tΔi_→j_,k_ :=
  Δ[$U, $k] /. {(v : τ | η | α | ξ)1 → vi,
    (v : t | T | y | a | x)1 → vj, (v : t | T | y | a | x)2 → vk};
```

```
tΔ1→1,2
```

$\mathbb{E}\Big[(a_1+a_2)\,\alpha_1 + (t_1+t_2)\,\tau_1,\ y_1\,\eta_1 + T_1\,y_2\,\eta_1 + x_1\,\xi_1 + x_2\,\xi_1,$

$1 + \frac{1}{2}\,\big(-2\,\hbar\,a_1\,T_1\,y_2\,\eta_1 + \gamma\,\hbar\,T_1\,y_1\,y_2\,\eta_1^2 - 2\,\hbar\,a_1\,x_2\,\xi_1 + \gamma\,\hbar\,x_1\,x_2\,\xi_1^2\big)\,\epsilon +$
$O[\epsilon]^2\Big]$

The Faddeev-Quesne formula:

```
𝕖q_,k_[x_] := e^(Sum_{j=1}^{k+1} (1 - q)^j x^j / (j (1 - q^j))); 𝕖q_[x_] := 𝕖q,$k[x]
```

```
R[QU, kk_] :=
  R[QU, kk] = 𝔼[-ℏ a2 t1/γ, ℏ x2 y1,
    Series[e^(ℏ γ^-1 t1 a2 - ℏ y1 x2)
      (e^(ℏ b1 a2) 𝕖qℏ,kk[ℏ y1 x2] /. b1 → γ^-1 (ϵ a1 - t1)),
     {ϵ, 0, kk}]];
tRi_,j_ :=
  R[$U, $k] /. {(v : t | T | y | a | x)1 → vi,
     (v : t | T | y | a | x)2 → vj};
t̄Ri_,j_ := t̄Ri,j = tRi,j ~Bj~ tSj;
```

```
{tR1,2, t̄R1,2}
```

$\Big\{\mathbb{E}\Big[-\frac{\hbar\,a_2\,t_1}{\gamma},\ \hbar\,x_2\,y_1,\ 1 + \Big(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2\Big)\,\epsilon + O[\epsilon]^2\Big],$

$\mathbb{E}\Big[\frac{\hbar\,a_2\,t_1}{\gamma},\ -\frac{\hbar\,x_2\,y_1}{T_1},\ 1 + \frac{1}{4\,\gamma\,T_1^2}$
$\big(-4\,\hbar\,a_1\,a_2\,T_1^2 - 4\,\gamma\,\hbar^2\,a_1\,T_1\,x_2\,y_1 - 4\,\gamma\,\hbar^2\,a_2\,T_1\,x_2\,y_1 - 3\,\gamma^2\,\hbar^3\,x_2^2\,y_1^2\big)$
$\epsilon + O[\epsilon]^2\Big]\Big\}$

tC is the counterclockwise spinner; t̄C is its inverse.

```
tCi_ := 𝔼[0, 0, Ti^(1/2) e^(-ϵ ai ℏ) + 0$k];
t̄Ci_ := 𝔼[0, 0, Ti^(-1/2) e^(ϵ ai ℏ) + 0$k];
```

```
Block[{$k = 3}, {tC1, t̄C2}]
```

$\Big\{\mathbb{E}\Big[0,\ 0,$
$\sqrt{T_1} - \hbar\,a_1\,\sqrt{T_1}\,\epsilon + \frac{1}{2}\,\hbar^2\,a_1^2\,\sqrt{T_1}\,\epsilon^2 - \frac{1}{6}\,\big(\hbar^3\,a_1^3\,\sqrt{T_1}\big)\,\epsilon^3 + O[\epsilon]^4\Big],$

$\mathbb{E}\Big[0,\ 0,\ \frac{1}{\sqrt{T_2}} + \frac{\hbar\,a_2\,\epsilon}{\sqrt{T_2}} + \frac{\hbar^2\,a_2^2\,\epsilon^2}{2\,\sqrt{T_2}} + \frac{\hbar^3\,a_2^3\,\epsilon^3}{6\,\sqrt{T_2}} + O[\epsilon]^4\Big]\Big\}$

```
Kink[QU, kk_] :=
  Kink[QU, kk] =
   Block[{$k = kk}, (tR1,3 t̄C2) ~B1,2~ tm1,2→1 ~B1,3~ tm1,3→1];
tKinki_ := Kink[$U, $k] /. {(v : t | T | y | a | x)1 → vi};
K̄ink[QU, kk_] :=
  K̄ink[QU, kk] =
   Block[{$k = kk}, (t̄R1,3 tC2) ~B1,2~ tm1,2→1 ~B1,3~ tm1,3→1];
t̄Kinki_ := K̄ink[$U, $k] /. {(v : t | T | y | a | x)1 → vi}
```

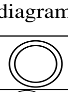
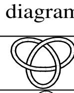
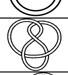
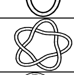
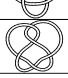
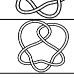
Alternative Algorithms

```
λalt,k_[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
    eq = ρ@e^(ξ xCU).ρ@e^(η yCU) == ρ@e^(d yCU).ρ@e^(c (t 1CU - 2 ϵ aCU)).ρ@e^(b xCU);
    {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /.
      C@1 → 0;
    Series[e^(-η y - ξ x + η ξ t + c t + d y - 2 ϵ c a + b x) /. so, {ϵ, 0, k}]]];
```

The Trefoil

```
Block[{$k = 1},
  Z = tR1,5 tR6,2 tR3,7 t̄C4 t̄Kink8 t̄Kink9 t̄Kink10;
  Do[Z = Z ~B1,k~ tm1,k→1, {k, 2, 10}]; Z]
```

$\mathbb{E}\Big[0,\ 0,\ \frac{T_1}{1 - T_1 + T_1^2} +$
$\big(\big(-2\,\hbar\,a_1\,T_1 - \gamma\,\hbar\,T_1^2 + 2\,\hbar\,a_1\,T_1^2 + 2\,\gamma\,\hbar\,T_1^3 - 3\,\gamma\,\hbar\,T_1^4 - 2\,\hbar\,a_1\,T_1^4 +$
$2\,\gamma\,\hbar\,T_1^5 + 2\,\hbar\,a_1\,T_1^5 - 2\,\gamma\,\hbar^2\,T_1\,x_1\,y_1 - 2\,\gamma\,\hbar^2\,T_1^4\,x_1\,y_1\big)\,\epsilon\big)\,/$
$\big(1 - 3\,T_1 + 6\,T_1^2 - 7\,T_1^3 + 6\,T_1^4 - 3\,T_1^5 + T_1^6\big) + O[\epsilon]^2\Big]$

diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon; unknotting number / amphicheiral	diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon; unknotting number / amphicheiral
	0_1^a / 0	1	0 / ✔; 0 / ✔		3_1^a / t	$t-1$	1 / ✘; 1 / ✘
	4_1^a / 0	$3-t$	1 / ✘; 1 / ✔		5_1^a / $2t^3+3t$	t^2-t+1	2 / ✘; 2 / ✘
	5_2^a / $5t-4$	$2t-3$	1 / ✘; 1 / ✘		6_1^a / $t-4$	$5-2t$	1 / ✔; 1 / ✘

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/