The PBW Problem. In $\mathcal{U}(\mathfrak{g}^\epsilon)$, bring $Z = y^3a^2x^2 \cdot y^2a^2x$ to yax-order. In other words, find $g \in \mathbb{Z}[\epsilon, t, y, a, x]$ such that $Z = \mathbb{O}(f = y_1^3 y_2^2 a_1^2 a_2^2 x_1^2 x_2 \colon y_1a_1x_1y_2a_2x_2) = \mathbb{O}(g\colon yax)$.

Solution, Part 1. In $\hat{\mathcal{U}}(\mathfrak{g}^\epsilon)$ we have

$$X_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2} := \mathbb{e}^{\tau_1 t}\mathbb{e}^{\eta_1 y}\mathbb{e}^{\alpha_1 a}\mathbb{e}^{\xi_1 x}\mathbb{e}^{\tau_2 t}\mathbb{e}^{\eta_2 y}\mathbb{e}^{\alpha_2 a}\mathbb{e}^{\xi_2 x} = \mathbb{e}^{\tau t}\mathbb{e}^{\eta y}\mathbb{e}^{\alpha a}\mathbb{e}^{\xi x} =: Y_{\tau,\eta,\alpha,\xi},$$

where τ, η, α, ξ are ugly functions of $\tau_1, \eta_i, \alpha_i, \xi_i$:

$$\tau = \tau_1 + \tau_2 - \frac{\log(1-\epsilon\eta_2\xi_1)}{\epsilon} = \tau_1 + \tau_2 + \eta_2\xi_1 + \frac{\epsilon}{2}\eta_2^2\xi_1^2 + \ldots,$$

$$\eta = \eta_1 + \frac{\mathbb{e}^{-\alpha_1}\eta_2}{(1-\epsilon\eta_2\xi_1)} = \eta_1 + \mathbb{e}^{-\alpha_1}\eta_2 + \epsilon\mathbb{e}^{-\alpha_1}\eta_2^2\xi_1 + \ldots,$$

$$\alpha = \alpha_1 + \alpha_2 + 2\log(1-\epsilon\eta_2\xi_1) = \alpha_1 + \alpha_2 - 2\epsilon\eta_2\xi_1 + \ldots,$$

$$\xi = \frac{\mathbb{e}^{-\alpha_2}\xi_1}{(1-\epsilon\eta_2\xi_1)} + \xi_2 = \mathbb{e}^{-\alpha_2}\xi_1 + \xi_2 + \epsilon\mathbb{e}^{-\alpha_2}\eta_2\xi_1^2 + \ldots.$$

Note 1. This defines a mapping $\Phi\colon \mathbb{R}^8_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2} \to \mathbb{R}^4_{\tau,\eta,\alpha,\xi}$.

Proof. $\mathfrak{g}^\epsilon$ has a 2D representation ρ:

```
ρt = (1 0; 0 1); ρy = (0 0; -ϵ 0);
ρa = ((1 + 1/ϵ)/2  0; 0  -(1 - 1/ϵ)/2); ρx = (0 1; 0 0);
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy,
  ρx.ρy - ρy.ρx == ρt - 2 ϵ ρa}

{True, True, True}
```

It is enough to verify the desired identity in ρ:

```
ME = MatrixExp;
Simplify[
 ME[τ1 ρt].ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[τ2 ρt].
    ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
   ME[τ0 ρt].ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx] /.
  {τ0 → -Log[1-ϵ η2 ξ1]/ϵ + τ1 + τ2, η0 → η1 + e^-α1 η2/(1-ϵ η2 ξ1),
   α0 → 2 Log[1 - ϵ η2 ξ1] + α1 + α2, ξ0 → e^-α2 ξ1/(1-ϵ η2 ξ1) + ξ2}]

True
```

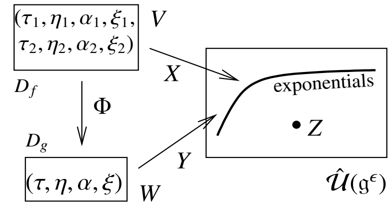
Solution, Part 2. But now, with $D_f = f(z \mapsto \partial_\zeta) = \partial^3_{\eta_1}\partial^2_{\alpha_1}\partial^2_{\xi_1}\partial^2_{\eta_2}\partial^2_{\alpha_2}\partial_{\xi_2}$,

$$Z = D_f X_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2}\Big|_{vs=0} = D_f Y_{\tau,\eta,\alpha,\xi}\Big|_{vs=0} = \mathbb{O}\left(D_f \mathbb{e}^{\tau t}\mathbb{e}^{\eta y}\mathbb{e}^{\alpha a}\mathbb{e}^{\xi x}\Big|_{vs=0} : yax\right) = \mathbb{O}(g\colon yax) :$$

```
Expand[∂{η1,3} ∂{α1,2} ∂{ξ1,2} ∂{η2,2} ∂{α2,2} ∂{ξ2,1} Exp[
     (-Log[1-ϵ η2 ξ1]/ϵ + τ1 + τ2) t + (η1 + e^-α1 η2/(1-ϵ η2 ξ1)) y +
      (2 Log[1 - ϵ η2 ξ1] + α1 + α2) a + (e^-α2 ξ1/(1-ϵ η2 ξ1) + ξ2) x
    ] /. (τ | η | α | ξ)1|2 → 0]

2 a^4 t^2 x y^3 + 4 t x^2 y^4 - 16 a t x^2 y^4 + 24 a^2 t x^2 y^4 - 16 a^3 t x^2 y^4 +
 4 a^4 t x^2 y^4 + 16 x^3 y^5 - 32 a x^3 y^5 + 24 a^2 x^3 y^5 - 8 a^3 x^3 y^5 + a^4 x^3 y^5 +
 2 a^4 t x y^3 ϵ - 8 a^5 t x y^3 ϵ + 8 x^2 y^4 ϵ - 40 a x^2 y^4 ϵ + 80 a^2 x^2 y^4 ϵ -
 80 a^3 x^2 y^4 ϵ + 40 a^4 x^2 y^4 ϵ - 8 a^5 x^2 y^4 ϵ - 4 a^5 x y^3 ϵ^2 + 8 a^6 x y^3 ϵ^2
```

Note 2. Replacing $f \to D_f$ (and likewise $g \to D_g$), we find that $D_g = \Phi_* D_f$.



Note 3. The two great evils of mathematics are non-commutativity and non-linearity. We traded one for the other.

Note 4. We could have done similarly with $\mathbb{e}^{\tau_1 t}\mathbb{e}^{\eta_1 y}\mathbb{e}^{\alpha_1 a}\mathbb{e}^{\xi_1 x} = \mathbb{e}^{\tau t + \eta y + \alpha a + \xi x}$, and with $S(\mathbb{e}^{\tau_1 t}\mathbb{e}^{\eta_1 y}\mathbb{e}^{\alpha_1 a}\mathbb{e}^{\xi_1 x})$, $\Delta(\mathbb{e}^{\tau_1 t}\mathbb{e}^{\eta_1 y}\mathbb{e}^{\alpha_1 a}\mathbb{e}^{\xi_1 x})$, $\prod_{i=1}^5 \mathbb{e}^{\tau_i t}\mathbb{e}^{\eta_i y}\mathbb{e}^{\alpha_i a}\mathbb{e}^{\xi_i x}$.

Fact. $R_{12} \to \exp\left(\partial_{\tau_1}\partial_{\alpha_2} + \partial_{y_1}\partial_{x_2}\right)(1 + \sum_{d\geq 1}\epsilon^d p_d)$, where the p_d are computable polynomials of a-priori bounded degrees.

Moral. We need to understand the pushforwards via maps like Φ of (formally ∞-order) "differential operators at 0", that in themselves are perturbed Gaussians. This turns out to be the same problem as "0-dimensional QFT" (except no integration is ever needed), and if $\epsilon^{k+1} = 0$, it is explicitly soluble.

References.

[BN] D. Bar-Natan, *Polynomial Time Knot Polynomial,* research proposal for the 2017 Killam Fellowship, ωεβ/K17.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* arXiv:1708.04853.

[BV2] D. Bar-Natan and R. van der Veen, *Poly-Time Knot Polynomials Via Solvable Approximations,* in preparation.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

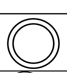
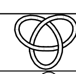
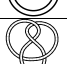
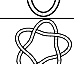
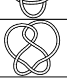
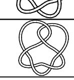
[Vo] H. Vo, University of Toronto Ph.D. thesis, in preparation.

dog·ma (dôg′mə, dŏg′-) The Free Dictionary, ωεβ/TFD

n. pl. **dog·mas** or **dog·ma·ta** (-mə-tə)

1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.

2. A principle or statement of ideas, or a group of such principles or statements, especially when considered to be authoritative or accepted uncritically: *"Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry" (Bertrand Russell).*

diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral
	0_1^a 0	1	0 / ✔ 0 / ✔		3_1^a t	$t-1$	1 / ✘ 1 / ✘
	4_1^a 0	$3-t$	1 / ✘ 1 / ✔		5_1^a $2t^3+3t$	t^2-t+1	2 / ✘ 2 / ✘
	5_2^a $5t-4$	$2t-3$	1 / ✘ 1 / ✘		6_1^a $t-4$	$5-2t$	1 / ✔ 1 / ✘

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1708/