

**Warning.** Conventions on this page change randomly from line to line.

**$Z^{w/2}$ .** The GGA story is about  $Z^{w/2}: \mathcal{K} \rightarrow \mathcal{A}^{w/2}$ , defined on arrows  $a$  by  $\pm a \mapsto \exp(\pm a)$ :

$$\text{Diagram: } \sum_{k,l,m \geq 0} \frac{(+)^k (-)^l (+)^m}{k! l! m!}$$

Where the target space  $\mathcal{A}^{w/2}$  is the space of unsigned arrow diagrams modulo

$$\begin{aligned} \text{TC: } & \quad \text{Diagram: } = 0 \\ \text{CH: } & \quad \text{Diagram: } = \text{Diagram: } - \text{Diagram: } \\ \text{and } \vec{4T}: & \quad \text{Diagram: } + \text{Diagram: } = 0 \end{aligned}$$

( $Z^{w/2}$  is a reduction of the much-studied  $Z^w$  [BND, BN]).

**The Euler Trick.** How best do non-commutative algebra with exponentials? Logarithms are from hell as  $e^f e^g = e^{\text{bch}(f,g)}$ , but Euler's from heaven: Let  $E$  be the derivation  $Ef := (\deg f)f$  ( $= xf$ , in  $\mathbb{Q}[[x]]$ ) and let  $\tilde{E}Z := Z^{-1}EZ$  ( $= x(\log Z)'$  in same). If  $\deg x = 1$  then  $\tilde{E}e^x = x$  and if  $F = e^f$  and  $G = e^g$ , then  $\tilde{E}(FG)$  is

$$(FG)^{-1}((EF)G + F(EG)) = G^{-1}(\tilde{E}F)G + \tilde{E}G = e^{-\text{ad } g}(\tilde{E}F) + \tilde{E}G.$$

**Scatter and Glow.** Apply  $\tilde{E}$  to  $Z(K)$ .  $EZ$  is shown:

$$\text{Diagram: } - \text{Diagram: } + \text{Diagram: }$$

**Tail scattering.** The algebra  $\mathbb{Q}[[b_i]]\langle a_{ij} \rangle$  modulo  $[a_{ij}, a_{kl}] = 0$  (loc),  $[a_{ij}, a_{ik}] = 0$  (TC), and  $[a_{ik}, a_{jk}] = -[a_{ij}, a_{jk}] = b_j a_{ik} - b_i a_{jk}$  (CH and  $\vec{4T}$ ), acts on  $V = \mathbb{Q}[[b_i]]\langle x_i = a_{i\infty} \rangle$  by  $[a_{ij}, x_i] = 0$ ,  $[a_{ij}, x_j] = b_i x_j - b_j x_i$ . Hence  $e^{\text{ad } a_{ij}} x_i = x_i$ ,  $e^{\text{ad } a_{ij}} x_j = e^{b_i} x_j + \frac{b_j}{b_i} (1 - e^{b_i}) x_i$ . Renaming  $\bar{x}_i = x_i/b_i$ ,  $T_i = e^{b_i}$ , get

$$[e^{\text{ad } a_{ij}}]_{\bar{x}_i, \bar{x}_j} = \begin{pmatrix} 1 & 1 - T_i \\ 0 & T_i \end{pmatrix}. \text{ Alternatively,}$$

$$\text{Diagram: } \begin{array}{c} y_i \\ \downarrow \\ x_i \end{array} \xrightarrow{\quad} \begin{array}{c} \bar{y}_i = \bar{x}_i \\ \bar{y}_j = (1 - T_i)\bar{x}_i + T_i \bar{x}_j \end{array}$$

**Linear Control Theory.**

If  $\begin{pmatrix} y \\ y_n \end{pmatrix} = \begin{pmatrix} \Xi & \phi \\ \theta & \alpha \end{pmatrix} \begin{pmatrix} x \\ x_n \end{pmatrix}$ , and we further impose  $x_n = y_n$ , then  $y = Bx$  where  $B = \Xi + \frac{\phi\theta}{1 - \alpha}$ . This fully explains the Gassner formulas and the GGA formula!

$$\text{Diagram: } M \rightarrow M \boxed{B} = B$$

All that remains now is to replace TC by something more interesting: with  $\epsilon^2 = 0$ ,

$$[a_{ij}, a_{ik}] = \epsilon(c_j a_{ik} - c_k a_{ij}).$$

Many further changes are also necessary, and the algebra is a lot more complicated and revolves around “quantization of Lie bialgebras” [EK, En]. But the spirit is right.

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