

### Loading, initializing variables, setting default degree to 6.

(The Mathematica packages FreeLie' and AwCalculus' are at  $\omega\beta/\text{WKO4}$ ).

```
path = "C:/drorbn/AcademicPensieve/";  
SetDirectory[path <> "2015-08/LesDiablerets-1508"];  
Get[path <> "Projects/WKO4/FreeLie.m"];  
Get[path <> "Projects/WKO4/AwCalculus.m"];  
x = LW@"x"; y = LW@"y"; u = LW@"u";  
$SeriesShowDegree = 6;
```

FreeLie' implements / extends  
 $\{\star, +, **, \text{ad}, \text{Ad}, \text{adSeries}, \text{AllCyclicWords},$   
 $\text{AllLyndonWords}, \text{AllWords}, \text{Arbitrator}, \text{ASeries}, \text{AW}, \text{b}, \text{BCH}, \text{BooleanSequence},$   
 $\text{BracketForm}, \text{BS}, \text{CC}, \text{Crop}, \text{cw}, \text{CW}, \text{CWS}, \text{CWSeries}, \text{D}, \text{Deg}, \text{DegreeScale},$   
 $\text{DerivationSeries}, \text{div}, \text{DK}, \text{DKS}, \text{DKSeries}, \text{EulerE}, \text{Exp}, \text{Inverse}, \text{j}, \text{J}, \text{JA},$   
 $\text{LieDerivation}, \text{LieMorphism}, \text{LieSeries}, \text{LS}, \text{LW}, \text{LyndonFactorization}, \text{Morphism},$   
 $\text{New}, \text{RandomCWSeries}, \text{Randomizer}, \text{RandomLieSeries}, \text{RC}, \text{SeriesSolve}, \text{Support}, \text{tb},$   
 $\text{TopBracketForm}, \text{tr}, \text{UndeterminedCoefficients}, \text{cMap}, \Gamma, \cup, \Delta, \sigma, \hbar, \rightarrow, \rightarrow\}$ .

FreeLie' is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 150814.

AwCalculus' implements / extends  
 $\{\star, +, \text{dA}, \text{dc}, \text{deg}, \text{dm}, \text{dA}, \text{dD}, \text{dS}, \text{dV}, \text{El}, \text{Es}, \text{hA}, \text{hm}, \text{hS}, \text{hD}, \text{hN},$   
 $\text{hS}, \text{RandomElSeries}, \text{RandomEsSeries}, \text{tA}, \text{tha}, \text{tm}, \text{tS}, \text{tD}, \text{tN}, \text{to}, \Gamma, \Delta\}.$

AwCalculus' is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 150814.

### BCH[x, y] (\* Can raise degree to 22 \*)

$$\begin{aligned} & \text{LS}\left[\overline{x+y}, \frac{\overline{xy}}{2}, \frac{1}{12}\overline{x\overline{xy}} + \frac{1}{12}\overline{\overline{xy}y}, \frac{1}{24}\overline{x\overline{xy}}, \right. \\ & - \frac{1}{720}\overline{x\overline{x\overline{xy}}} + \frac{1}{180}\overline{x\overline{xy}y} + \frac{1}{180}\overline{x\overline{xy}y} + \frac{1}{120}\overline{\overline{xy}\overline{xy}} + \\ & \frac{1}{360}\overline{x\overline{xy}}\overline{xy} - \frac{1}{720}\overline{\overline{xy}y}\overline{yy}, - \frac{\overline{xxx}\overline{xy}}{1440} + \frac{1}{360}\overline{x\overline{xy}}\overline{yy} + \\ & \left. \frac{1}{240}\overline{x\overline{xy}}\overline{xy} + \frac{1}{720}\overline{x\overline{xy}}\overline{xy} - \frac{\overline{xy}\overline{yy}}{1440}, \dots \right] \end{aligned}$$

### KV Direct.

```
{F = LS[{x, y}], Fs, G = LS[{x, y}, Gs]]; Fs["y"] = 1/2;  
SeriesSolve[{F, G},  
 $\hbar^{-1} (\text{LS}[x+y] - \text{BCH}[y, x]) \equiv F - G - \text{Ad}[-x][F] + \text{Ad}[y][G]$ ] //  
divx[F] + divy[G]  $\equiv$   
 $\frac{1}{2} \text{tr}_u[\text{adSeries}\left[\frac{\text{ad}}{\text{e}^{\text{ad}-1}}, x\right][u] + \text{adSeries}\left[\frac{\text{ad}}{\text{e}^{\text{ad}-1}}, y\right][u] -$   
 $\text{adSeries}\left[\frac{\text{ad}}{\text{e}^{\text{ad}-1}}, \text{BCH}[x, y]\right][u]];$ 
```

### {F, G} (\* Can raise degree to 13 \*)

$$\begin{aligned} & \text{LS}\left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{xy}y} + \frac{1}{360}\overline{\overline{xy}y}, \right. \\ & - \frac{1}{720}\overline{x\overline{x\overline{xy}}} + \frac{1}{240}\overline{x\overline{xy}y} + \frac{1}{240}\overline{\overline{xy}\overline{xy}} + \frac{1}{720}\overline{x\overline{xy}}\overline{xy} - \\ & \frac{\overline{xy}\overline{yy}}{1440}, - \frac{\overline{xxx}\overline{xy}}{5040} - \frac{\overline{xxx}\overline{xy}}{1344} + \frac{13\overline{xx}\overline{xy}}{15120} + \frac{1}{840}\overline{x\overline{xy}}\overline{xy} + \\ & \frac{\overline{xx}\overline{xy}\overline{xy}}{3360} + \frac{\overline{xy}\overline{xy}\overline{yy}}{6720} + \frac{\overline{xy}\overline{xy}\overline{yy}}{1260} + \frac{\overline{xy}\overline{xy}\overline{yy}}{1680} - \frac{\overline{xy}\overline{yy}}{10080}, \dots \Big], \\ & \text{LS}\left[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{x\overline{xy}}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{xy}y} + \frac{1}{180}\overline{\overline{xy}y}, \right. \\ & - \frac{1}{720}\overline{x\overline{x\overline{xy}}} + \frac{1}{240}\overline{x\overline{xy}y} + \frac{1}{240}\overline{\overline{xy}\overline{xy}} + \frac{1}{720}\overline{x\overline{xy}}\overline{xy} - \\ & \frac{\overline{xy}\overline{yy}}{1440}, - \frac{\overline{xxx}\overline{xy}}{10080} - \frac{\overline{xxx}\overline{xy}}{2016} + \frac{\overline{xx}\overline{xy}\overline{yy}}{1890} + \frac{\overline{xx}\overline{xy}\overline{yy}}{1120} + \frac{\overline{xx}\overline{xy}\overline{yy}}{5040} + \\ & \left. \frac{\overline{xy}\overline{yy}\overline{yy}}{2520} + \frac{1}{840}\overline{x\overline{xy}}\overline{xy} + \frac{\overline{xy}\overline{xy}\overline{xy}}{1260} - \frac{\overline{xy}\overline{yy}\overline{yy}}{5040}, \dots \right] \end{aligned}$$

### Meaningless calculations.

$\{\text{b}[F, G], \text{tr}_x[F]\}$

$$\begin{aligned} & \left\{ \text{LS}\left[0, 0, -\frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{48}\overline{\overline{xy}y}, \frac{1}{720}\overline{x\overline{\overline{xy}y}} - \frac{1}{240}\overline{x\overline{\overline{xy}y}} - \right. \right. \\ & \frac{\overline{\overline{xy}\overline{xy}y}}{1440} - \frac{1}{720}\overline{x\overline{xy}}\overline{xy} - \frac{1}{360}\overline{\overline{xy}y}\overline{yy}, \frac{\overline{xx}\overline{\overline{xy}y}}{1440} - \\ & \frac{1}{480}\overline{x\overline{xy}}\overline{yy} - \frac{1}{288}\overline{\overline{xy}\overline{xy}y} - \frac{7\overline{xx}\overline{xy}\overline{xy}}{2880} + \frac{\overline{xy}\overline{yy}\overline{yy}}{2880}, \dots \Big], \right. \\ & \left. \text{CWS}\left[-\frac{V}{6}, \frac{\overline{yy}}{24}, \frac{\overline{xx}\overline{y}}{180} + \frac{\overline{xy}\overline{y}}{80} - \frac{\overline{yy}\overline{y}}{360}, -\frac{\overline{xx}\overline{yy}}{180} + \frac{\overline{xy}\overline{xy}}{240} - \frac{\overline{xy}\overline{yy}}{240} - \right. \right. \\ & \frac{\overline{xxx}\overline{y}}{5040} + \frac{\overline{xxx}\overline{yy}}{6720} - \frac{\overline{xx}\overline{xy}}{1120} + \frac{945}{945} + \frac{336}{336} + \frac{\overline{xy}\overline{yy}}{6720} + \frac{\overline{yy}\overline{yy}}{10080}, \\ & \left. \frac{\overline{xxxx}\overline{y}}{3360} - \frac{\overline{xxx}\overline{yy}}{1344} - \frac{\overline{xx}\overline{xy}\overline{yy}}{2240} + \frac{13\overline{xx}\overline{xy}\overline{yy}}{2016} + \frac{\overline{xy}\overline{yy}\overline{yy}}{10080} - \right. \\ & \left. \frac{\overline{xy}\overline{yy}\overline{yy}}{3780} - \frac{\overline{xy}\overline{xy}\overline{yy}}{840} + \frac{\overline{xy}\overline{yy}\overline{yy}}{5040} + \frac{\overline{xy}\overline{yy}\overline{yy}}{2240} + \frac{\overline{yy}\overline{yy}\overline{yy}}{6720} + \frac{\overline{yy}\overline{yy}\overline{yy}}{60480}, \dots \right] \} \end{aligned}$$

(Also implemented:  $\partial_\lambda$  and derivations in general, tb,  $e^{\partial_\lambda}$  and morphisms in general, div, j, Drinfel'd-Kohno, etc.)

### The [BND] "vertex" equations.

```
 $\alpha = \text{LS}[\{x, y\}, \text{as}]; \beta = \text{LS}[\{x, y\}, \text{bs}];$   
 $\gamma = \text{CWS}[\{x, y\}, \text{ys}];$   
 $\nu = \text{Es}[\langle x \rightarrow \alpha, y \rightarrow \beta \rangle, \gamma];$   
 $\kappa = \text{CWS}[\{x\}, \text{xs}]; \text{Cap} = \text{Es}[\langle x \rightarrow \text{LS}[0], \kappa \rangle];$   
 $\text{Rs}[a_-, b_-] := \text{Es}[\langle a \rightarrow \text{LS}[0], b \rightarrow \text{LS}[LW@a] \rangle, \text{CWS}[0]]; \text{R4Eqn} = \nu \star \star (\text{Rs}[x, z] // \text{d}\Delta[x, x, y]) \equiv \text{Rs}[y, z] \star \star \text{Rs}[x, z] \star \star \nu;$   
 $\text{UnitarityEqn} =$   
 $(\nu \star \star (\nu // \text{dA})) \equiv \text{Es}[\langle x \rightarrow \text{LS}[0], y \rightarrow \text{LS}[0] \rangle, \text{CWS}[0]]);$   
 $\text{CapEqn} = (\nu \star \star (\text{Cap} // \text{d}\Delta[x, x, y])) // \text{dc}[x] // \text{dc}[y]) =$   
 $(\text{Cap}(\text{Cap} // \text{d}\sigma[x, y])) // \text{dc}[x] // \text{dc}[y]);$   
 $\beta s["x"] = 1/2; \beta s["y"] = 0;$   
 $\text{SeriesSolve}[\{\alpha, \beta, \gamma, \kappa\},$   
 $(\text{n}^\perp \text{R4Eqn}) \wedge \text{UnitarityEqn} \wedge \text{CapEqn}];$   
 $\{\nu, \kappa\}$ 
```

SeriesSolve::ArbitrarilySetting: In degree 1 arbitrarily setting {xs[x]  $\rightarrow$  0}.

SeriesSolve::ArbitrarilySetting: In degree 3 arbitrarily setting {as[x, y]  $\rightarrow$  0}.

SeriesSolve::ArbitrarilySetting: In degree 5 arbitrarily setting {as[x, x, y]  $\rightarrow$  0}.

General::stop:

Further output SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. >>

$$\begin{aligned} & \left\{ \text{Es}\left[\left\langle \overline{x} \rightarrow \text{LS}\left[0, -\frac{\overline{xy}}{24}, 0, \frac{7\overline{xx}\overline{xy}}{5760} - \frac{7\overline{xy}\overline{yy}}{5760} + \frac{\overline{xy}\overline{yy}}{1440}, 0, \right. \right. \right. \right. \right. \\ & \left. \left. \left. \left. \left. \left. - \frac{31\overline{xxx}\overline{xy}}{967680} + \frac{31\overline{xxx}\overline{yy}}{483840} - \frac{83\overline{xx}\overline{yy}\overline{y}}{967680} - \frac{31\overline{xy}\overline{xy}\overline{yy}}{725760} - \frac{31\overline{x}\overline{xy}\overline{xy}\overline{y}}{645120} + \right. \right. \right. \right. \right. \\ & \left. \left. \left. \left. \left. \left. + \frac{13\overline{xx}\overline{yy}\overline{yy}}{241920} + \frac{101\overline{xy}\overline{xy}\overline{yy}}{1451520} + \frac{527\overline{xy}\overline{xy}\overline{xy}}{5806080} - \frac{\overline{xy}\overline{yy}\overline{yy}}{60480}, \dots \right. \right. \right. \right. \right] \right], \\ & \overline{y} \rightarrow \text{LS}\left[\frac{\overline{y}}{2}, -\frac{\overline{xy}}{12}, 0, \frac{\overline{xx}\overline{xy}}{5760} - \frac{1}{720}\overline{x\overline{xy}} + \frac{1}{720}\overline{\overline{xy}y}, -\frac{\overline{xx}\overline{xy}}{7680} + \right. \\ & \frac{\overline{xx}\overline{xy}\overline{y}}{3840} - \frac{\overline{xy}\overline{xy}\overline{y}}{6912} - \frac{\overline{xx}\overline{xy}\overline{y}}{645120} + \frac{23\overline{xxx}\overline{xy}\overline{y}}{483840} - \frac{13\overline{xx}\overline{xy}\overline{y}}{161280} - \frac{\overline{xx}\overline{xy}\overline{y}}{22680} - \\ & \frac{41\overline{xx}\overline{xy}\overline{xy}}{580608} + \frac{\overline{xy}\overline{xy}\overline{yy}}{15120} + \frac{\overline{xy}\overline{xy}\overline{yy}}{12096} + \frac{71\overline{xy}\overline{xy}\overline{xy}}{483840} - \frac{\overline{xy}\overline{yy}\overline{yy}}{30240}, \dots \Big], \\ & \text{CWS}\left[0, -\frac{\overline{xy}}{48}, 0, \frac{\overline{xx}\overline{y}}{2880} + \frac{\overline{xy}\overline{y}}{2880} + \frac{\overline{xy}\overline{yy}}{5760} + \frac{\overline{yy}\overline{yy}}{2880}, 0, \right. \\ & \left. - \frac{\overline{xxx}\overline{y}}{120960} - \frac{\overline{xxx}\overline{yy}}{120960} - \frac{\overline{xx}\overline{xy}\overline{y}}{120960} - \frac{\overline{xx}\overline{yy}\overline{y}}{120960} - \frac{\overline{xy}\overline{xy}\overline{yy}}{241920} - \frac{\overline{xy}\overline{yy}\overline{yy}}{120960} - \right. \\ & \left. \frac{\overline{xy}\overline{yy}\overline{yy}}{120960} - \frac{\overline{yy}\overline{yy}\overline{yy}}{120960}, \dots \right], \\ & \text{CWS}\left[0, -\frac{\overline{xx}}{96}, 0, \frac{\overline{xxx}}{11520}, 0, -\frac{\overline{xxxx}}{725760}, \dots \right] \} \end{aligned}$$

Video and more at <http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/>