Expansions Day 5: Back to 4D

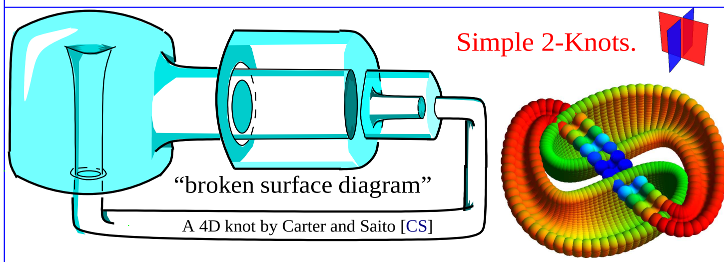
Abstract. We will repeat the 3D story of the previous 3 talks one dimension up, in 4D. Surprisingly, there's more room in 4D, and things get easier, at least when we restrict our attention to "w-knots", or to "simply-knotted 2-knots". But even then there are intricacies, and we try to go beyond simply-knotted, we are completely confused.

Recall.

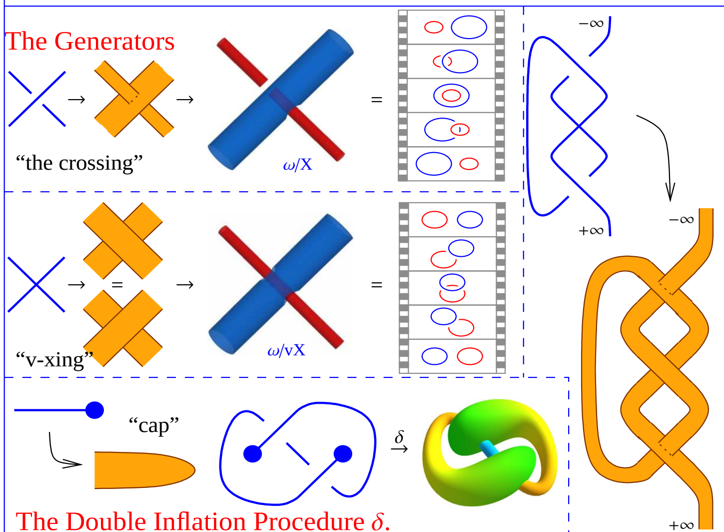
$\mathcal{K}$ Topology $\xrightarrow{Z\text{: high algebra}}$ $\mathcal{A} := \operatorname{gr}\mathcal{K}$ Combinatorics $\xrightarrow{\text{given a "Lie" algebra } \mathfrak{g}}$ "$\mathcal{U}(\mathfrak{g})$"

solving finitely many equations in finitely many unknowns — low algebra: pictures represent formulas

Simple 2-Knots.

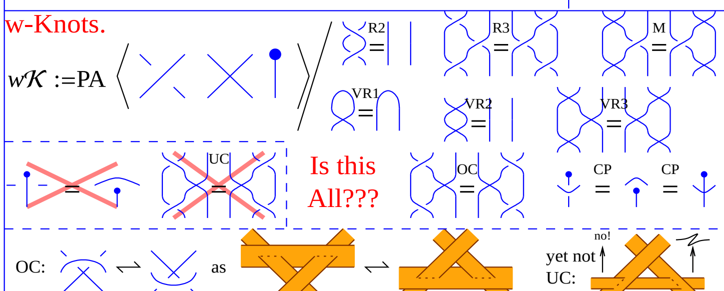


The Generators



The Double Inflation Procedure δ.

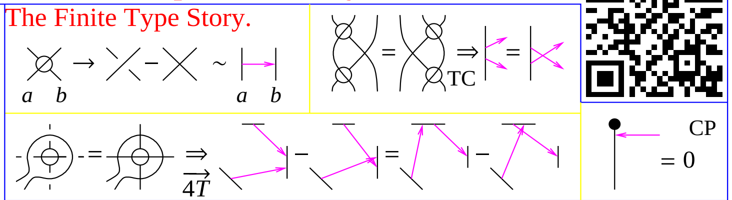
w-Knots.



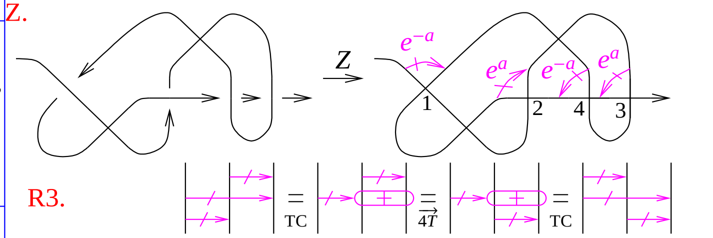
A Big Open Problem.

δ maps w-knots onto simple 2-knots. To what extent is it a bijection? What other relations are required? In other words, **find a simple description of simple 2-knots**. Kawauchi [Ka] may already know the answer.

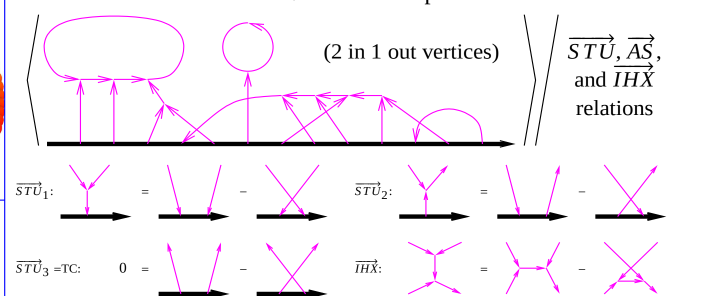
The Finite Type Story.



Z.



The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to



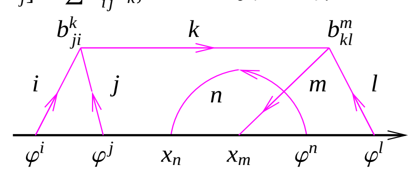
Corollaries. (1) Only wheels and isolated arrows persist:

$$\mathcal{A}^w(\uparrow_n) \cong \mathcal{U}(FL(n)^n_{tb} \ltimes CW(n)) \quad\text{and}\quad \zeta := \log Z \in FL(n)^n \times CW(n)$$

has completely explicit formulas using natural FL/CW operations [BN].
(2) Related to f.d. Lie algebras!

Low Algebra.

With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij} x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij} b^m_{kl} \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

Differential Ops.

We can also interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\text{Fun}(\mathfrak{g})$: $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator, and $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\operatorname{ad} x$: $(x\varphi)(y) := \varphi([x, y])$.

Too easy so far!

Yet once you add "foam vertices", it gets related to the Kashiwara-Vergne problem [KV] as told by Alekseev-Torossian [AT]: