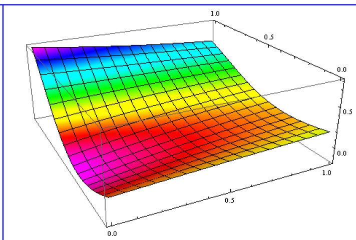
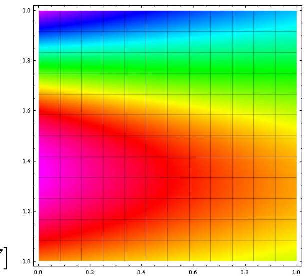
Dessert: Hilbert's 13th Problem, in Full Colour

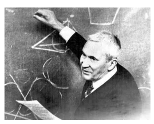
Dror Bar−Natan, Toronto November 2014. More at http://www.math.toronto.edu/~drorbn/Talks/Fields-1411

Abstract. To break a week of deep thinking with a nice colourful light dessert, we will present the Kolmogorov-Arnold solution of Hilbert's 13th problem with lots of computer-generated rainbow-painted 3D pictures.
In short, Hilbert asked if a certain specific function of three variables can be written as a multiple (yet finite) composition of continuous functions of just two variables. Kolmogorov and Arnold showed him silly (ok, it took about 60 years, so it was a bit tricky) by showing that **any** continuous function f of any finite number of variables is a finite composition of continuous functions of a single variable and several instances of the binary function "+" (addition). For $f(x,y)=xy$, this may be $xy=\exp(\log x+\log y)$. For $f(x,y,z)=x^y/z$, this may be $\exp(\exp(\log y+\log\log x)+(-\log z))$. What might it be for (say) the real part of the Riemann zeta function?
The only original material in this talk will be the pictures; the math was known since around 1957.

$\frac{1}{3}\,\mathrm{Re}(\zeta(x+iy))$ on $[0,1]\times[13,17]$

Hilbert



Kolmogorov

Arnold (by Moser)

Fix an irrational $\lambda>0$, say $\lambda=(\sqrt{5}-1)/2$. All functions are continuous.

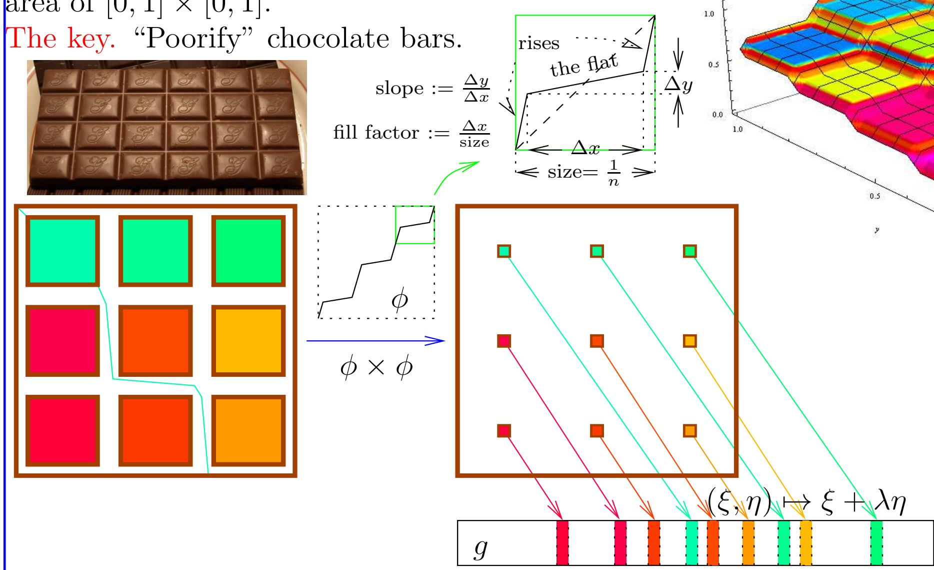
Theorem. There exist five $\phi_i:[0,1]\to[0,1]$ ($1\le i\le 5$) so that for every $f:[0,1]\times[0,1]\to\mathbb{R}$ there exists a $g:[0,1+\lambda]\to\mathbb{R}$ so that

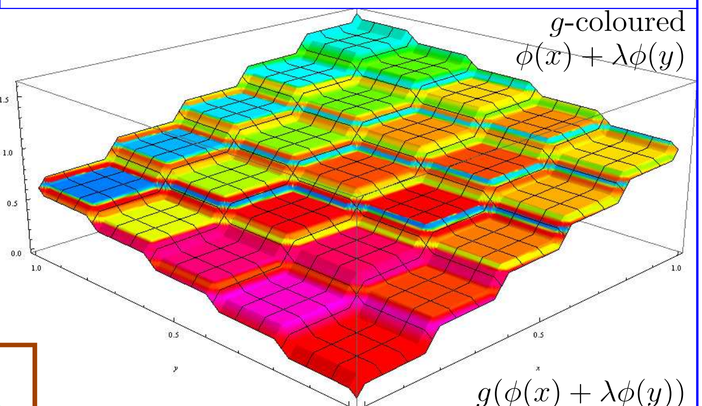
$$f(x,y)=\sum_{i=1}^{5} g(\phi_i(x)+\lambda\phi_i(y))$$

for every $x,y\in[0,1]$.

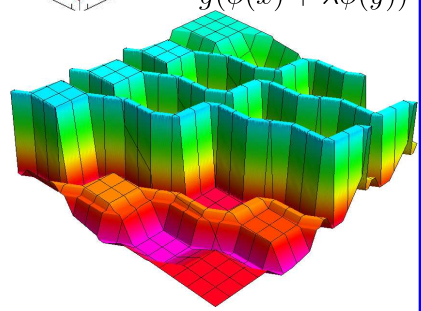
Step 1. If $\epsilon>0$ and $f:[0,1]\times[0,1]\to\mathbb{R}$, then there exists $\phi:[0,1]\to[0,1]$ and $g:[0,1+\lambda]\to\mathbb{R}$ so that $|f(x,y)-g(\phi(x)+\lambda\phi(y))|<\epsilon$ on at least 98% of the area of $[0,1]\times[0,1]$.

The key. "Poorify" chocolate bars.

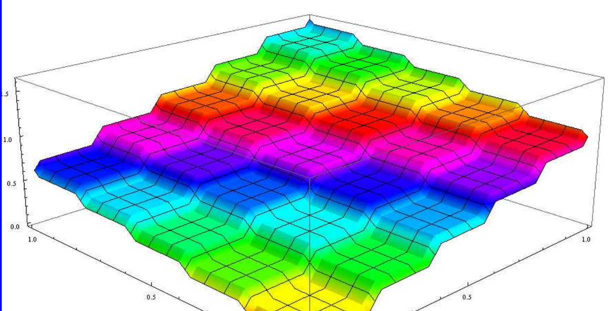




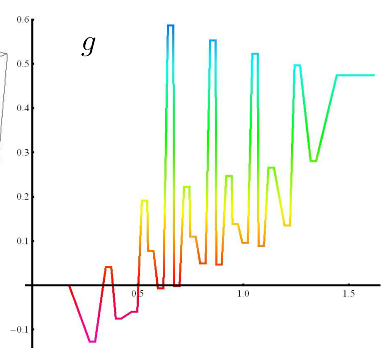
g-coloured $\phi(x)+\lambda\phi(y)$

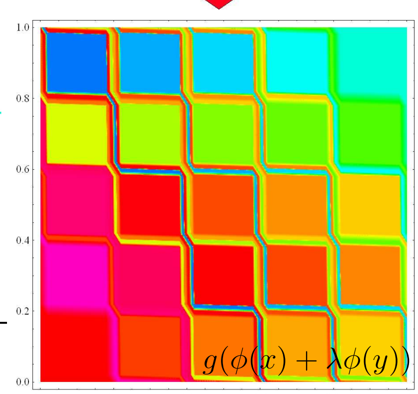


$g(\phi(x)+\lambda\phi(y))$



z-coloured' $\phi(x)+\lambda\phi(y)$





$g(\phi(x)+\lambda\phi(y))$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Fields-1411/