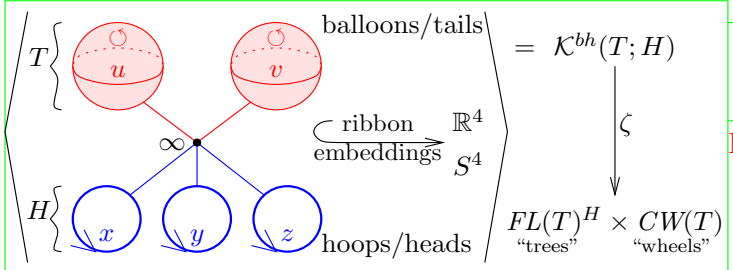
Finite Type Invariants of Ribbon Knotted Balloons and Hoops

Abstract. On my September 17 Geneva talk (ω/sep) I described a certain trees-and-wheels-valued invariant ζ of ribbon knotted loops and 2-spheres in 4-space, and my October 8 Geneva talk (ω/oct) describes its reduction to the Alexander polynomial. Today I will explain how that same invariant arises completely naturally within the theory of finite type invariants of ribbon knotted loops and 2-spheres in 4-space.

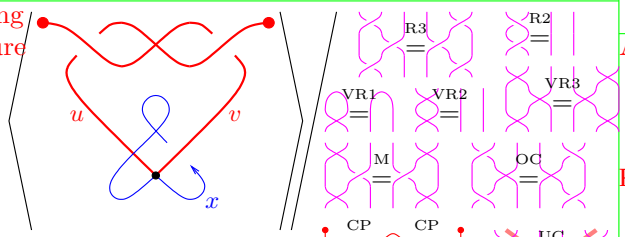


My goal is to tell you why such an invariant is expected, yet not to derive the computable formulas.

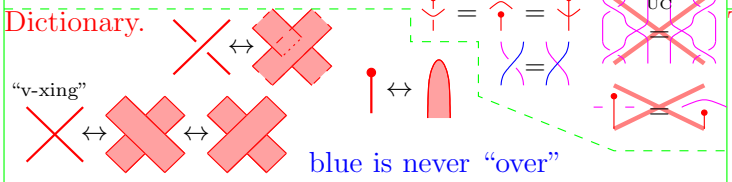
Disturbing Conjecture

$\mathcal{K}^{bh} = \mathbb{Q}$


Satoh



Dictionary.



Expansions

the semi-virtual := − i.e. − or −

Let $\mathcal{I}^n := \langle \text{pictures with} \geq n \text{ semi-virts} \rangle \subset \mathcal{K}^{bh}$.
We seek an "expansion"

$$Z : \mathcal{K}^{bh} \to \operatorname{gr} \mathcal{K}^{bh} = \widehat{\bigoplus} \mathcal{I}^n/\mathcal{I}^{n+1} =: \mathcal{A}^{bh}$$

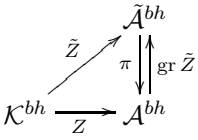
satisfying "property U": if $\gamma \in \mathcal{I}^n$, then

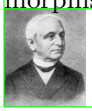
$$Z(\gamma) = (0, \dots, 0, \gamma/\mathcal{I}^{n+1}, *, *, \dots).$$


X.-S. Lin

Why? • Just because, and this is vastly more general. • $(\mathcal{K}^{bh}/\mathcal{I}^{n+1})^\star$ is "finite-type/polynomial invariants". • The Taylor example: Take $\mathcal{K} = C^\infty(\mathbb{R}^n)$, $\mathcal{I} = \{f \in \mathcal{K} : f(0) = 0\}$. Then $\mathcal{I}^n = \{f : f \text{ vanishes like } |x|^n\}$ so $\mathcal{I}^n/\mathcal{I}^{n+1}$ is homogeneous polynomials of degree n and Z is a "Taylor expansion"! (So Taylor expansions are vastly more general than you'd think).

Plan. We'll construct a graded $\tilde{\mathcal{A}}^{bh}$, a surjective graded $\pi : \tilde{\mathcal{A}}^{bh} \to \mathcal{A}^{bh}$, and a filtered $\tilde{Z} : \mathcal{K}^{bh} \to \mathcal{A}^{bh}$ so that $\pi /\!\!/ \operatorname{gr} \tilde{Z} = Id$ (property U: if $\deg D = n$, $\tilde{Z}(\pi(D)) = \pi(D) + (\deg \geq n)$). Hence • π is an isomorphism. • $Z := \tilde{Z} /\!\!/ \pi$ is an expansion.




"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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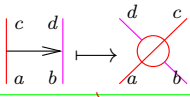
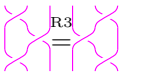
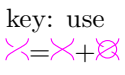
Action 1.

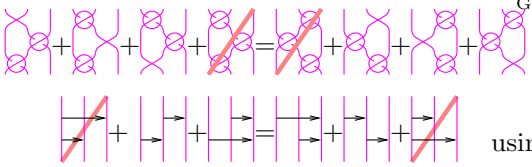
$\tilde{\mathcal{A}}^{bh} = \mathbb{Q}$ ⟨ T, H ⟩ / (degree=# of arrows)

π :  (then connect using xings or v-xings)

Deriving $\overrightarrow{4T}$.

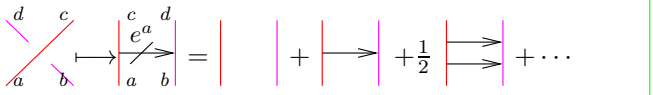
Start from  key: use 

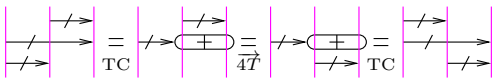

Goussarov-Polyak-Viro


in $\mathcal{I}^n/\mathcal{I}^{n+1}$

using TC □

Action 2.

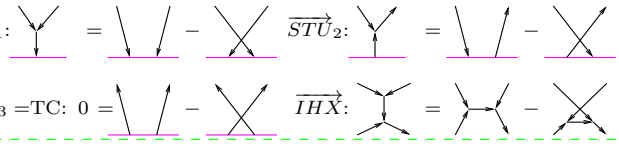
$\tilde{Z}$: 

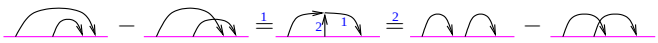
R3. 

Exercise. Prove property U.

The Bracket-Rise Theorem.

$\mathcal{A}^{bh} \cong$ ⟨ (2 in 1 out vertices) ⟩ / CP, B, $\overrightarrow{STU}$, $\overrightarrow{AS}$, and $\overrightarrow{IHX}$ relations



Proof. 

Corollaries. (1) Related to Lie algebras! (2) Only trees and wheels persist.

Theorem. $\mathcal{A}^{bh}$ is a bi-algebra. The space of its primitives is $FL(T)^H \times CW(T)$, and $\zeta = \log Z$.

ζ is computable! ζ of the Borromean tangle, to degree 5: