Balloons and Hoops and their Universal Finite−Type Invariant, BF Theory, and an Ultimate Alexander Invariant

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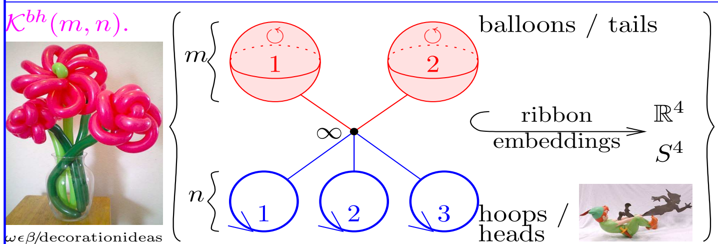
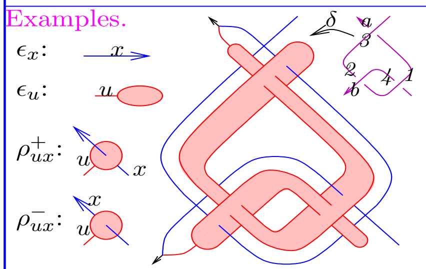
$\omega\epsilon\beta$:=http://www.math.toronto.edu/~drorbn/Talks/Oxford-130121

Scheme. • Balloons and hoops in $\mathbb{R}^4$, algebraic structure and relations with 3D.
• An ansatz for a "homomorphic" invariant: computable, related to finite-type and to BF.
• Reduction to an "ultimate Alexander invariant".

$\mathcal{K}^{bh}(m,n)$.



Examples. **I mean business!**

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T0 = Rm[3, a] Rp[b, 2] Rp[1, 4];
ζ = T0 // dm[2, 1, 1] // dm[4, b, b] //
    dm[1, a, a] // dm[3, a, a];
ζ[{5}] /. {w_LW :→ (Deg[w] + 1) ! w,
   w_CW :→ Deg[w] ! w}

μ[CWS[-[a], -2 [ab], -3 [aab] - 3 [abb],
   -4 [aaab] + 42 [aabb] - 60 [abab] - 4 [abbb],
   -5 [aaaab] + 110 [aaabb] - 180 [aabab] +
    110 [aabbb] - 180 [ababb] - 5 [abbbb]],
  h[b] LS[2 ⟨a⟩, 0, -24 ⟨aab⟩,
     -60 ⟨aaab⟩ + 60 ⟨aabb⟩, -120 ⟨aaaab⟩ +
      900 ⟨aaabb⟩ + 360 ⟨aabab⟩ - 120 ⟨aabbb⟩] +
   h[a] LS[-2 ⟨a⟩ + 2 ⟨b⟩, 9 ⟨ab⟩, 26 ⟨aab⟩ -
      26 ⟨abb⟩, 60 ⟨aaab⟩ - 255 ⟨aabb⟩ + 60 ⟨abbb⟩,
     119 ⟨aaaab⟩ - 1504 ⟨aaabb⟩ + 118 ⟨aabab⟩ +
      1504 ⟨aabbb⟩ + 1386 ⟨ababb⟩ - 119 ⟨abbbb⟩]]
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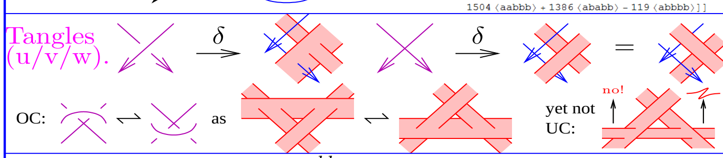
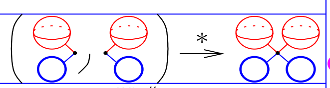
Tangles (u/v/w).



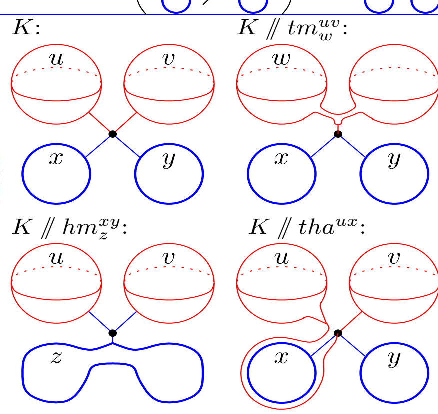
• δ injects u-Knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
• δ maps v/w-tangles map to $\mathcal{K}^{bh}$; the kernel contains Reidemeister moves and the "overcrossings commute" relation, and conjecturally, that's all. Allowing punctures and cuts, δ is onto.

Operations Punctures & Cuts | Connected Sums.



Meta-Group-Action. If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

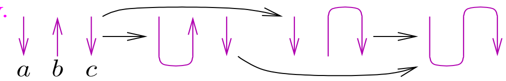
"MGA"



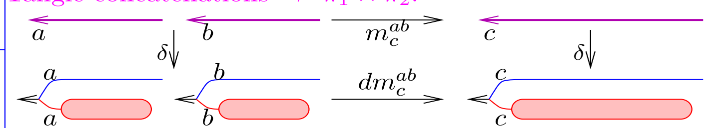
("$/\!\!/$" is newspeak for "apply an operator" and for "composition left to right")

Properties.
• Associativities: $m^{ab}_a \,/\!\!/\, m^{ac}_a = m^{bc}_b \,/\!\!/\, m^{ab}_a$, for $m = tm, hm$.
• Action axiom t: $tm^{uv}_w \,/\!\!/\, tha^{wx} = tha^{ux} \,/\!\!/\, tha^{vx} \,/\!\!/\, tm^{uv}_w$.
• Action axiom h: $hm^{xy}_z \,/\!\!/\, tha^{uz} = tha^{ux} \,/\!\!/\, tha^{uy} \,/\!\!/\, hm^{xy}_z$.
• SD Product: $dm^{ab}_c := tha^{ab} \,/\!\!/\, tm^{ab}_c \,/\!\!/\, hm^{ab}_c$ is associative.

Meta-associativity. $m^{ab}_a \,/\!\!/\, m^{ac}_a = m^{bc}_b \,/\!\!/\, m^{ab}_a$.

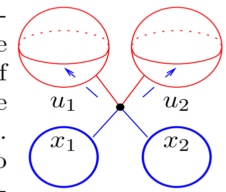


Tangle concatenations → $\pi_1 \ltimes \pi_2$.



Thus we seek homomorphic invariants of $\mathcal{K}^{bh}$!

Invariant #0. With Π_1 denoting "honest π_1", map $\gamma \in \mathcal{K}^{bh}(m,n)$ to the triple $(\Pi_1(\gamma^c), (u_i), (x_j))$, where the meridian of the balls u_i normally generate Π_1, and the "longtitudes" x_j are some elements of Π_1. $*$ acts like $*$, tm acts by "merging" two meridians/generators, hm acts by multiplying two longtitudes, and tha^{ux} acts by "conjugating a meridian by a longtitude":



Not computable! (but nearly)

$$(\Pi, (u, \ldots), (x, \ldots)) \mapsto (\Pi * \langle \bar{u} \rangle / (u = x\bar{u}x^{-1}), (\bar{u}, \ldots), (x, \ldots))$$

Failure #0. Can we write the x's as free words in the u's? If $x = uv$, compute $x \,/\!\!/\, tha^{ux}$:

$$x = uv \to \bar{u}v = u^x v = u^{\bar{u}v} v = u^{u^x v} v = u^{u^{u^x v} v} v = \cdots$$

The Meta-Group-Action M. Let T be a set of "tail labels" ("balloon colours"), and H a set of "head labels" ("hoop colours"). Let $FL = FL(T)$ and $FA = FA(T)$ be the (completed graded) free Lie and free associative algebras on generators T and let $CW = CW(T)$ be the (completed graded) vector space of cyclic words on T, so there's $\operatorname{tr} : FA \to CW$. Let $M(T,H) := \{(\bar{\lambda} = (x : \lambda_x)_{x \in H}; \omega) : \lambda_x \in FL, \omega \in CW\}$

$$= \left\{ \left(x : \underset{}{\overset{u\ \ v}{\curlyvee}}, \ y : \overset{v}{\Big|} - \tfrac{22}{7} \overset{u\ \ \ v}{\curlyvee} ; \ \bigcirc \right) \ldots \right\}$$

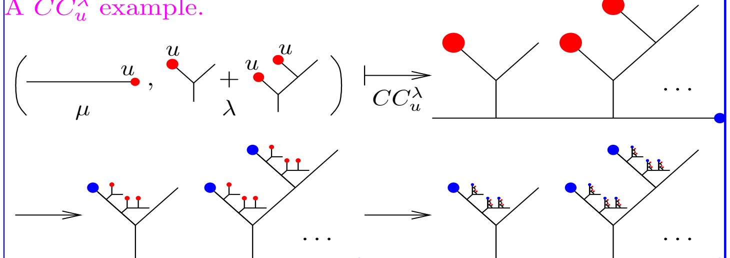
Operations. Set $(\bar{\lambda}_1; \omega_1) * (\bar{\lambda}_2; \omega_2) := (\bar{\lambda}_1 \cup \bar{\lambda}_2; \omega_1 + \omega_2)$ and with $\mu = (\bar{\lambda}; \omega)$ define

$$tm^{uv}_w : \mu \mapsto \mu \,/\!\!/\, (u, v \mapsto w),$$

$$hm^{xy}_z : \mu \mapsto \left(\left(\ldots, \widehat{x : \lambda_x}, \widehat{y : \lambda_y}, \ldots, z : \operatorname{bch}(\lambda_x, \lambda_y)\right); \omega\right)$$

$$tha^{ux} : \mu \mapsto \underbrace{\mu \overset{\text{"stable apply"}}{\overbrace{/\!\!/\!\!/}} (u \mapsto e^{\operatorname{ad}\lambda_x}(\bar{u})) \,/\!\!/\, (\bar{u} \mapsto u)}_{\mu /\!\!/ CC^{\lambda_x}_u} + \underbrace{(0; J_u(\lambda_x))}_{\text{the "}J\text{-spice"}}$$

A CC^λ_u example.



Video and more at http://www.math.toronto.edu/~drorbn/Talks/Oxford-130121/ and at http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/