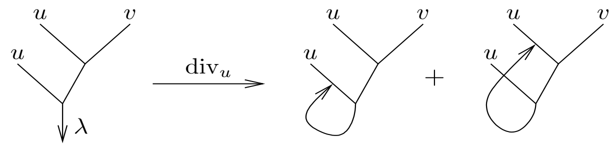
Balloons and Hoops and their Universal Finite−Type Invariant, 2

The Meta-Cocycle J. Set $J_u(\lambda) := J(1)$ where

$$J(0) = 0, \qquad \lambda_s = \lambda \,/\!/\, CC_u^{s\lambda},$$

$$\frac{dJ(s)}{ds} = (J(s) \,/\!/\, \mathrm{der}(u \mapsto [\lambda_s, u])) + \mathrm{div}_u\, \lambda_s,$$

and where $\mathrm{div}_u\, \lambda := \mathrm{tr}(u\sigma_u(\lambda))$, $\sigma_u(v) := \delta_{uv}$, $\sigma_u([\lambda_1, \lambda_2]) := \iota(\lambda_1)\sigma_u(\lambda_2) - \iota(\lambda_2)\sigma_u(\lambda_1)$ and ι is the inclusion $FL \hookrightarrow FA$:



Claim. $CC_u^{\mathrm{bch}(\lambda_1,\lambda_2)} = CC_u^{\lambda_1} \,/\!/\, CC_u^{\lambda_2 /\!/ CC_u^{\lambda_1}}$ and

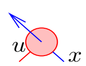
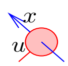
$$J_u(\mathrm{bch}(\lambda_1, \lambda_2)) = J_u(\lambda_1) \,/\!/\, CC_u^{\lambda_2 /\!/ CC_u^{\lambda_1}} + J_u(\lambda_2 \,/\!/\, CC_u^{\lambda_1}),$$

and hence tm, hm, and tha form a meta-group-action.

Why ODEs? **Q.** Find f s.t. $f(x+y) = f(x)f(y)$. **A.** $\frac{df(s)}{ds} = \frac{d}{d\epsilon} f(s+\epsilon) = \frac{d}{d\epsilon} f(s)f(\epsilon) = f(s)C$. Now solve this ODE using Picard's theorem or power series.

Alekseev Torossian

The Invariant ζ. Set $\zeta(\rho^\pm) = (\pm u_x; 0)$. This at least defines an invariant of u/v/w-tangles, and if the topologists will deliver a "Reidemeister" theorem, it is well defined on $\mathcal{K}^{bh}$.

ζ:  $\longmapsto \left(x : +\Big|^{u} ; 0\right)$  $\longmapsto \left(x : -\Big|^{u} ; 0\right)$

Theorem. ζ is (the log of) a universal finite type invariant (a homomorphic expansion) of w-tangles.

Tensorial Interpretation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \mathfrak{g})$ and $\tau : CW(T) \to \mathrm{Fun}(\oplus_T \mathfrak{g})$. Together, $\tau : M(T,H) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \oplus_H \mathfrak{g})$, and hence

$$e^\tau : M(T,H) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

ζ and BF Theory. Let A denote a $\mathfrak{g}$-connection on S^4 with curvature F_A, and B a $\mathfrak{g}^*$-valued 2-form on S^4. For a hoop γ_x, let $\mathrm{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x. For a ball γ_u, let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be the integral of B (transported via A to ∞) on γ_u.

Cattaneo

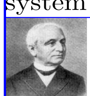
Loose Conjecture. For $\gamma \in \mathcal{K}(T,H)$,

$$\int \mathcal{D}A\mathcal{D}B e^{\int B \wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B))} \bigotimes_x \mathrm{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

Issues. How exactly is B transported via A to ∞? How does the ribbon condition arise? Or if it doesn't, could it be that ζ can be generalized??

The β quotient, 1. • Arises when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra.
• Arises when reducing by relations satisfied by the weight system of the Alexander polynomial.



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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Paper in progress: $\omega\epsilon\beta$/kbh

The β quotient, 2. Let $R = \mathbb{Q}[\![\{c_u\}_{u \in T}]\!]$ and $L_\beta := R \otimes T$ with central R and with $[u,v] = c_u v - c_v u$ for $u, v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,

$$\mu \to (\bar\lambda; \omega) \quad \text{with } \bar\lambda = \sum_{x \in H,\, u \in T} \lambda_{ux} ux, \quad \lambda_{ux}, \omega \in R,$$

$$\mathrm{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v} - 1}\left(\frac{e^{c_u}-1}{c_u}u + e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

if $\lambda = \sum \lambda_v v$ then with $c_\lambda := \sum \lambda_v c_v$,

$$u \,/\!/\, CC_u^\lambda = \left(1 + c_u\lambda_u \frac{e^{c_\lambda}-1}{c_\lambda}\right)^{-1} \left(e^{c_\lambda} u - c_u \frac{e^{c_\lambda}-1}{c_\lambda} \sum_{v \neq u} \lambda_v v\right),$$

$\mathrm{div}_u\, \lambda = c_u \lambda_u$, and the ODE for J integrates to

$$J_u(\lambda) = \log\left(1 + \frac{e^{c_\lambda}-1}{c_\lambda} c_u \lambda_u\right),$$

so ζ is formula-computable to all orders! Can we simplify?

Repackaging. Given $((x : \lambda_{ux}); \omega)$, set $c_x := \sum_v c_v \lambda_{vx}$, replace $\lambda_{ux} \to \alpha_{ux} := c_u \lambda_{ux} \frac{e^{c_x}-1}{c_x}$ and $\omega \to \log \omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get "β calculus".

β Calculus. Let $\beta(H,T)$ be

$$\left\{ \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \;\middle|\; \begin{array}{l} \omega \text{ and the } \alpha_{ux}\text{'s are} \\ \text{rational functions in} \\ \text{variables } t_u, \text{ one for} \\ \text{each } u \in T. \end{array} \right\},$$

In preparation, Selmani & B–N.

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$tha^{ux} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta - \gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1 + \alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v \neq u} \gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

On long knots, ω is the Alexander polynomial!

Why bother? (1) An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination!). *If there should be an Alexander invariant to have an algebraic categorification, it is this one!*

See also $\omega\epsilon\beta$/regina, $\omega\epsilon\beta$/gwu.

Why bother? (2) Related to A-T, K-V, and E-K, should have vast generalization beyond w-knots and the Alexander polynomial. See also $\omega\epsilon\beta$/wko, $\omega\epsilon\beta$/caen, $\omega\epsilon\beta$/swiss.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Oxford-130121/ and at http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/