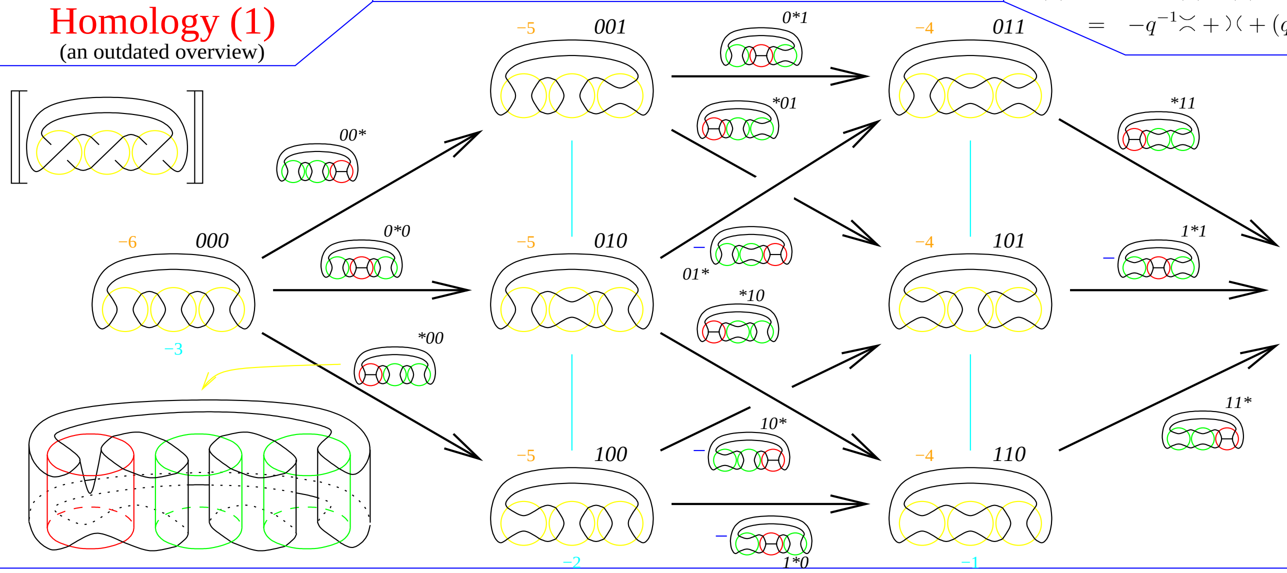
Local Khovanov Homology (1)

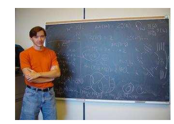
(an outdated overview)

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

$J : \overcrossing \mapsto q)(- q^2 \asymp, \quad J : \undercrossing \mapsto -q^{-2}\asymp + q^{-1})($

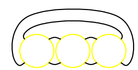
R2 $J : \ \mapsto -q^{-1}\ + \ + \ - q\ = -q^{-1}\asymp +)(+ (q+q^{-1})\asymp - q\asymp =)($

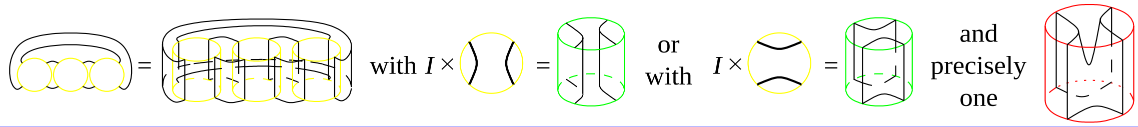




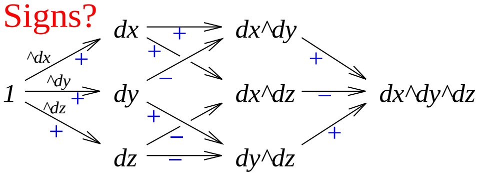
Mikhail Khovanov

What is it?

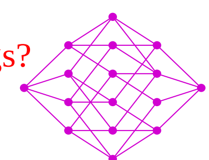
A cube for each knot/link projection; Vertices: All fillings of  with)(or with $\asymp$.

Edges: All fillings of $I\times$  with $I\times$)(= ... or with $I\times \asymp$ = ... and precisely one ... .

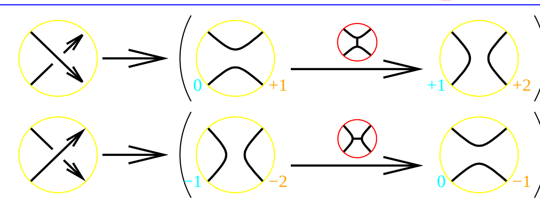
Signs?



More crossings?



General Crossings



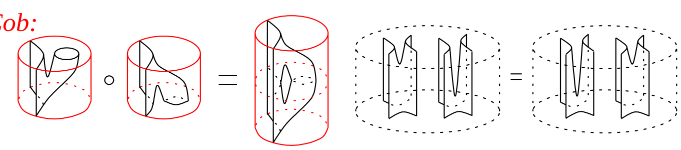
Where does it live?

In *Kom(Mat(<Cob> / {S, T, G, NC})) / homotopy*

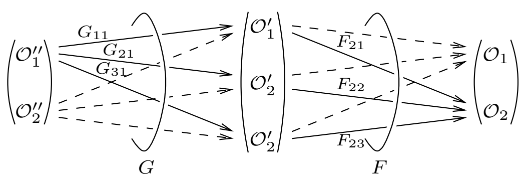
Kom: Complexes Mat: Matrices

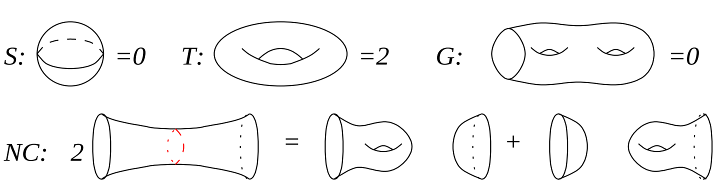
Cob: Cobordisms <...>: Formal lin. comb.

Cob:



Mat(C):





Complexes:

$$\Omega = \left(\Omega^{-n_-} \longrightarrow \Omega^{-n_-+1} \longrightarrow \cdots \qquad \longrightarrow \Omega^{n_+}\right)$$

Morphisms:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} & \longrightarrow & \cdots \\ & & \downarrow F^{r-1} & & \downarrow F^r & & \downarrow F^{r+1} & & \\ \cdots & \longrightarrow & \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} & \longrightarrow & \cdots \end{array}$$

Homotopies:

$$\begin{array}{ccccc} \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} \\ F^{r-1}\downarrow\downarrow G^{r-1} & \swarrow h^r & F^r\downarrow\downarrow G^r & \swarrow h^{r+1} & F^{r+1}\downarrow\downarrow G^{r+1} \\ \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} \end{array}$$

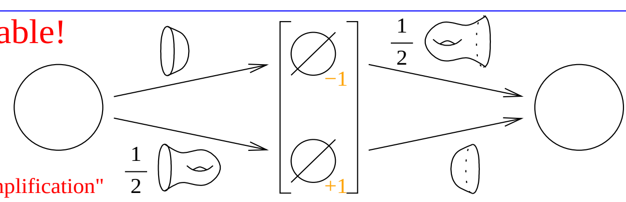
$$F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$$

All arrows in an arbitrary additive category!

The Main Point. “The cube”, Kh(L), is an up-to-homotopy invariant of knots and links. It's Euler characteristic is the Jones polynomial, yet it is strictly stronger than the Jones polynomial. It is functorial (in the appropriate sense) and practically computable.

Computable!

via "complex simplification"



The Categorification Speculative Paradigm. • Every object in math is the Euler characteristic of a complex.

• Every operation lifts to an operation between complexes.

• Every identity remains true, up to homotopy.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/