The Pure Virtual Braid Group is Quadratic[1]

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http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
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Abstract Generalities

Let K be a unital algebra over a field $\mathbb{F}$ with char $\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; so $K/I \xrightarrow[\epsilon]{\sim} \mathbb{F}$.

Definition. Say that K is quadratic if its associated graded $\operatorname{gr} K=\bigoplus_{p=0}^\infty I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=q(K)=\langle V=I/I^2\rangle/\langle R_2=\ker(\bar\mu_2: V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (q is a lovely functor). Then K is quadratic iff the obvious $\mu: A\to\operatorname{gr}K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

The Overall Strategy. Consider the "singularity tower" of (K,I) (here ":" means $\otimes_K$ and μ is (always) multiplication):

$$\cdots\ I^{:p+1}\xrightarrow{\mu_{p+1}} I^{:p}\xrightarrow{\mu_p} I^{:p-1}\longrightarrow\cdots\longrightarrow K$$

We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- What's $I^{:p}/\mu(I^{:p+1})$?
- How injective is this tower?

Lemma. $I^{:p}/\mu(I^{:p+1})\simeq(I/I^2)^{\otimes p}=V^{\otimes p}$; set $\pi: I^{:p}\to V^{\otimes p}$.

Flow Chart.

$$\begin{tikzcd}
\boxed{\text{Any }(K,I)} \arrow[r, "\text{Prop 1}"] & \boxed{\text{2-local}} \arrow[r, "\text{Prop 2}"] & \boxed{\text{Quadratic}} \\
\boxed{K=PvB_n} \arrow[r, "\text{Thm S}"', "\text{by Peter}"] & \boxed{\text{Hutchings' Criterion}} \arrow[r] & \boxed{\text{2-injective}}
\end{tikzcd}$$

Proposition 1. The sequence

$$\mathfrak{R}_p:=\bigoplus_{j=1}^{p-1}\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)\xrightarrow{\partial} I^{:p}\xrightarrow{\mu_p} I^{:p-1}$$

is exact, where $\mathfrak{R}_2:=\ker\mu: I^{:2}\to I$; so (K,I) is "2-local".

The Free Case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, define $\psi: F\to F$ by $x_i\mapsto x_i+\epsilon(x_i)$. Then $J_0:=\psi(J)$ is $\{w\in F:\deg w>0\}$. For J_0 it is easy to check that $\mathfrak{R}_2=\mathfrak{R}_p=0$, and hence the same is true for every J.

The General Case. If $K=F/\langle M\rangle$ (where M is a vector space of "moves") and $I\subset K$, then $I=J/\langle M\rangle$ where $J\subset F$. Then $I^{:p}=J^{:p}/\sum J^{:j-1}:\langle M\rangle:J^{:p-j}$ and we have

$$\begin{tikzcd}
J^{:p} \arrow[rr, "\mu_F", "1\text{-}1"'] \arrow[d, "\text{onto}"', "\pi_p"] & & J^{:p-1} \arrow[d, "\pi_{p-1}"', "\text{onto}"] \\
I^{:p}=J^{:p}/\sum J^{:}:\langle M\rangle:J^{:} \arrow[rr, "\mu"] & & I^{:p-1}=J^{:p-1}/\sum J^{:}:\langle M\rangle:J^{:}
\end{tikzcd}$$

So[2] $\ker(\mu)=\pi_p\left(\mu_F^{-1}(\ker\pi_{p-1})\right)=\pi_p\left(\sum\mu_F^{-1}\left(J^{:}:\langle M\rangle:J^{:}\right)\right)=\sum\pi_p\left(J^{:}:\mu_F^{-1}\langle M\rangle:J^{:}\right)=\sum I^{:}:\mathfrak{R}_2:I^{:}=:\sum_{j=1}^{p-1}\mathfrak{R}_{p,j}$.

$\mathfrak{R}_2$ is simpler than may seem! It's an "augmentation bimodule" ($I\mathfrak{R}_2=0=\mathfrak{R}_2 I$ thus $xr=\epsilon(x)r=r\epsilon(x)=rx$ for $x\in K$ and $r\in\mathfrak{R}_2$), and hence $\mathfrak{R}_2=\pi_2(\mu_F^{-1}M)$.

$$\begin{tikzcd}
J^{:2} \arrow[r, "\mu_F", "1\text{-}1"'] \arrow[d, "\pi_2"] & J\supset M \arrow[d, "\pi_1"] \\
I^{:2} \arrow[r, "\mu"] & I=J/\langle M\rangle
\end{tikzcd}$$

$\mathfrak{R}_p$ is simpler than may seem! In $\mathfrak{R}_{p,j}=I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}$ the I factors may be replaced by $V=I/I^2$. Hence

$$\mathfrak{R}_p\simeq\bigoplus_{j=1}^{p-1}V^{\oplus j-1}\otimes\pi_2(\mu_F^{-1}M)\otimes V^{\otimes p-j-1}.$$

Claim. $\pi(\mathfrak{R}_{p,j})=R_{p,j}$; namely,

$$\pi\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)=V^{\otimes j-1}\otimes R_2\otimes V^{\otimes p-j-1}.$$

Why Care?

• In abstract generality, $\operatorname{gr}K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z: K\to\hat A$, becomes wonderful mathematics:

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

2-Injectivity. A (one-sided infinite) sequence

$$\cdots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}} K_p\xrightarrow{\delta_p}\cdots\longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\cdots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\bar\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\bar\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots$$

is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

Proposition 2. If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}/\sum\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)$. But that's the degree p piece of $q(K)$.

The X Lemma (inspired by [Hut]).

$$\begin{tikzcd}
A_0 \arrow[dr, "\alpha_0"] & & C_0 \\
& B \arrow[ur, "\beta_0"] \arrow[dr, "\beta_1"] & \\
A_1 \arrow[ur, "\alpha_1"] & & C_1
\end{tikzcd}$$


Michael Hutchings
will compute homology for food

If the above diagram is Conway ($\asymp$) exact, then its two diagonals have the same "2-injectivity defect". That is, if $A_0\to B\to C_0$ and $A_1\to B\to C_1$ are exact, then $\ker(\beta_1\circ\alpha_0)/\ker\alpha_0\simeq\ker(\beta_0\circ\alpha_1)/\ker\alpha_1$.

Proof. $\frac{\ker(\beta_1\circ\alpha_0)}{\ker\alpha_0}\xrightarrow[\alpha_0]{\sim}\ker\beta_1\cap\operatorname{im}\alpha_0$

$$=\ker\beta_0\cap\operatorname{im}\alpha_1\xleftarrow[\alpha_1]{\sim}\frac{\ker(\beta_0\circ\alpha_1)}{\ker\alpha_1}.$$

The Hutchings Criterion [Hut]. The singularity tower of (K,I) is 2-injective iff on the right, $\ker(\pi\circ\partial)=\ker(\partial)$. That is, iff every "diagrammatic syzygy" is also a "topological syzygy".

$$\begin{tikzcd}
\mathfrak{R}_p \arrow[dr, "\partial"] & & I^{:p-1} \\
& I^{:p} \arrow[ur, "\mu_p"] \arrow[dr, "\pi"] & \\
I^{:p+1} \arrow[ur, "\mu_{p+1}"] & & V^{\otimes p}
\end{tikzcd}$$

Conclusion. We need to know that (K,I) is "syzygy complete" — that every diagrammatic syzygy is also a topological syzygy, that $\ker(\pi\circ\partial)=\ker(\partial)$.

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