Facts and Dreams About v–Knots and Etingof–Kazhdan, 1

This is an overview with too many and not enough details. I apologize.

Dror Bar–Natan at Swiss Knots 2011
http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/

Foots & refs on PDF version, page 3.

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan [EK] quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\mathrm{gr}_I K := \widehat{\bigoplus} I^m/I^{m+1}$ has a product μ, especially, $\mu_{11} : (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle \ker \mu_{11} \rangle$ of K surjects using μ on $\mathrm{gr}\, K$.
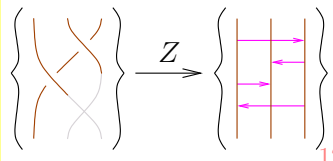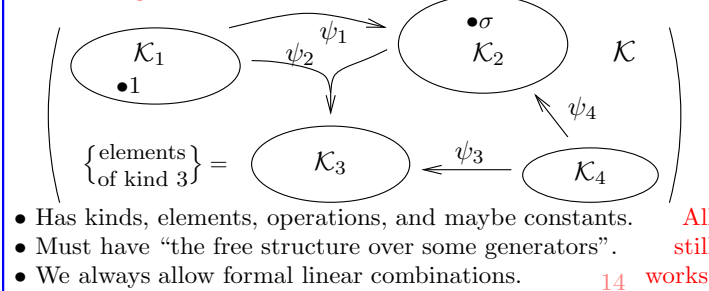
Peter Lee

The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filtered $Z : K \to \mathcal{A}$ for which $\mathrm{gr}\, Z : \mathrm{gr}\, K \to \mathcal{A}$ inverts μ.

Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way. 6

Example 2. For $K = \mathbb{Q}PvB_n =$ "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one. 12

General Algebraic Structures[1].

- Has kinds, elements, operations, and maybe constants.   All
- Must have "the free structure over some generators".   still
- We always allow formal linear combinations. 14   works!

**Example 3.** Quandle: a set K with an op $\wedge$ s.t.
$$1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$
$\mathcal{A}(K)$ is a graded Leibniz[2] algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:
$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$
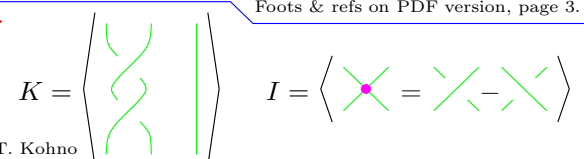15

Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an A_n-associator, and the Grothendieck-Teichmüller story[3] arises naturally.

GT   ASS   GRT

Drinfel'd [Dr1,2] 19
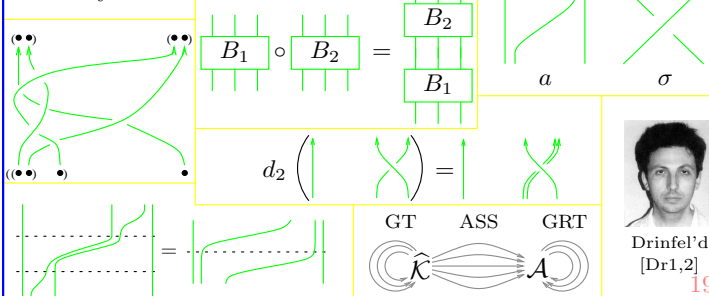
Example 1.

T. Kohno

$$K = \left\langle \;\; \right\rangle \qquad I = \left\langle \;\; = \;\; - \;\; \right\rangle$$

$$(K/I^{m+1})^\star = (\text{invariants of type } m) =: \mathcal{V}_m$$

$$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1} \quad C = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \left\langle \;\; \right\rangle$$

$$\ker \mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle \text{4T relations} \rangle$$

$$\mathcal{A} = A_n = \begin{pmatrix} \text{horizontal chord dia-} \\ \text{grams mod 4T} \end{pmatrix} = \left\langle \;\; \right\rangle \text{4T}$$

Z: universal finite type invariant, the Kontsevich integral. 9

Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients. 11
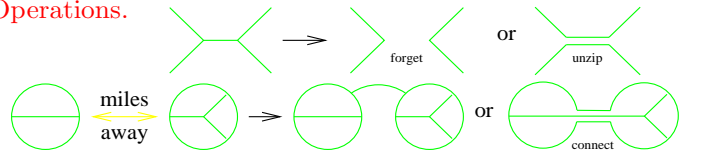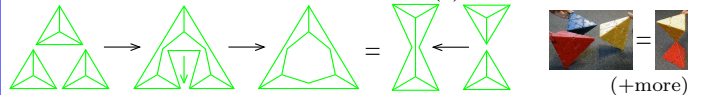
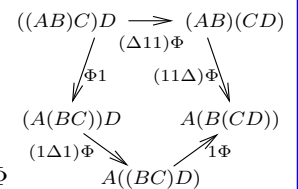**Example 5 - Knotted Trivalent Graphs.**

D. Thurston [Th]

Operations.

forget   unzip   or

miles away

connect   or
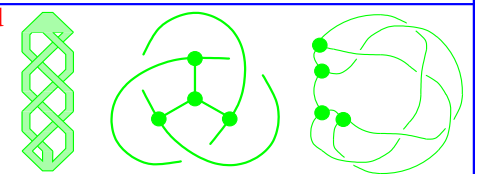
Presentation. KTG is generated by ribbon twists and the tetrahedron $\triangle$, modulo the relation(s):

(+more)

Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

$:= \Phi \in \mathcal{A}(\uparrow_3)$

A $\mathcal{U}(\mathfrak{g})$-Associator:

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",
$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

$((AB)C)D \xrightarrow{(\Delta 11)\Phi} (AB)(CD)$

$\Phi 1 \qquad (11\Delta)\Phi$

$(A(BC))D \qquad A(B(CD))$

$(1\Delta 1)\Phi$

$(1\Delta 1)\Phi \qquad 1\Phi$

$A((BC)D)$

$$\mathcal{A}(\uparrow_2) := \left\langle \;\; \right\rangle \Big/ \text{AS}, \;\; \xrightarrow[\text{metrized } \mathfrak{g}]{\text{Given a}} \mathcal{U}(\mathfrak{g})^{\otimes 2}$$

$$= \;\; - \;\;$$
$$= \;\; - \;\;$$

$(\deg = \frac{1}{2}\#\{\text{trivalent vertices}\})$ $\mathfrak{g} = \langle X_a \rangle$

Penrose

Cvitanovic

$$\longrightarrow \sum_{a,b,c,d,e,f=1}^{\dim \mathfrak{g}} f_{abc} f_{dce} X_a X_d X_f \otimes X_b X_f X_e$$
25

Video and more at http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/