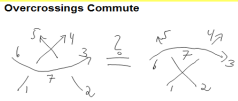
We Mean Business

```
 1         (* WP: Wedge Product *)
 2 WSort[expr_] := Expand[expr /. w_W :> Signature[w]*Sort[w]];
 3 WP[0, _] = WP[_, 0] = 0;
 4 WP[a_, b_] := WSort[Distribute[a ** b] /.
 5     (c1_. * w1_W) ** (c2_. * w2_W) :> c1 c2 Join[w1, w2]];
 6
 7         (* IM: Interior Multiplication *)
 8 IM[{}, expr_] := expr;
 9 IM[i_, w_W] := If[FreeQ[w, i], 0,
10     -(-1)^Position[w, i][[1,1]]*DeleteCases[w, i] ];
11 IM[{is___, i_}, w_W] := IM[{is}, IM[i, w]];
12 IM[is_List, expr_] := expr /. w_W :> IM[is, w]
13
14         (* pA on Crossings *)
15 pA[Xp[i_,j_,k_,l_]] := AHD[(t[i]==t[k])(t[j]==t[l]), {i,l}, W[j,k],
16     W[l,i] + (t[i]-1)W[l,j] - t[l]W[l,k] + W[i,j] + t[l]W[j,k] ];
17 pA[Xm[i_,j_,k_,l_]] := AHD[(t[i]==t[k])(t[j]==t[l]), {i,j}, W[k,l],
18     t[j]W[i,j] - t[j]W[i,l] + W[j,k] + (t[i]-1)W[j,l] + W[k,l] ]
19
20         (* Variable Equivalences *)
21 ReductionRules[Times[]] = {};
22 ReductionRules[Equal[a_, b__]] := (# -> a)& /@ {b};
23 ReductionRules[eqs_Times] := Join @@ (ReductionRules /@ List@@eqs)
24
25         (* AHD: Alexander Half Densities *)
26 AHD[eqs_, is_, -os_, p_] := AHD[eqs, is, os, Expand[-p]];
27 AHD /: Reduce[AHD[eqs_, is_, os_, p_]] :=
28   AHD[eqs, Sort[is], WSort[os], WSort[p /. ReductionRules[eqs]]];
29 AHD /: AHD[eqs1_,is1_,os1_,p1_] AHD[eqs2_,is2_,os2_,p2_] := Module[
30   {glued = Intersection[Union[is1, is2], List@@Union[os1, os2]]},
31   Reduce[AHD[
32     eqs1*eqs2 //. eq1_Equal*eq2_Equal /;
33       Intersection[List@@eq1, List@@eq2] =!= {} :> Union[eq1, eq2],
34     Complement[Union[is1, is2], glued],
35     IM[glued, WP[os1, os2]],
36     IM[glued, WP[p1, p2]]
37 ]] ]
38
39         (* pA on Circuit Diagrams *)
40 pA[cd_CircuitDiagram, eqs___] := pA[cd, {}, AHD[Times[eqs], {}, W[], W[]]];
41 pA[cd_CircuitDiagram, done_, ahd_AHD] := Module[
42   {pos = First[Ordering[Length[Complement[List @@ #, done]] & /@ cd]]},
43   pA[Delete[cd, pos], Union[done, List @@ cd[[pos]]], ahd*pA[cd[[pos]]]]
44 ];
45 pA[CircuitDiagram[], _, ahd_AHD] := ahd
```

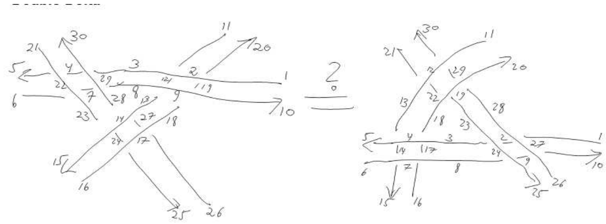
Comments online **2.** `W[i1,i2,...]` represents $i_1 \wedge i_2 \wedge \ldots$. To sort it we `Sort` its arguments and multiply by the `Signature` of the permutation used. **3.** The wedge product of 0 with anything is 0. **4-5.** The wedge product of two things involves applying the `Distribute`eive law, `Join`ing all pairs of `W`'s, and `WSort`ing the result. **8.** Inner multiplying by an empty list of indices does nothing. **9-10.** Inner multiplying a single index yields 0 if that index is not pressent, otherwise it's a sign and the index is deleted. **11-12.** Aftwrwards it's simple recursion. **15-18.** For the crossings `Xp` and `Xm` it is straightforward to determine the incoming strands, the outgoing ones, and the variable equivalences. The associated half-densities are just as in the formulas. **21-23.** The technicalities of imposing variable equivalences are annoying. **26.** That's all we need from the definition of a tensor product. **27-28.** Straightforward simplifications. **29.** The (circuit algebra) product of two Alexander Half Densities: **30.** The `glued` strands are the intersection of the ins and the outs. **32-33.** Merging the variable equivalences is tricky but natural. **34-35.** Removing the `glued` strands from the ins and outs. **36 The Key Point.** The wedge product of the half-densities, inner with the `glued` strands. **40-45.** A quick implementation of a "thin scanning" algorithm for multiple products. The key line is **42**, where we select the next crossing we multiply in to be the crossing with the fewest "loose strands".

▪ Overcrossings Commute



Hence "w-knots"

```
In[15]:= Equal[
          pA[CircuitDiagram[Xp[1, 7, 4, 6], Xp[2, 3, 5, 7]]],
          pA[CircuitDiagram[Xp[2, 7, 5, 6], Xp[1, 3, 4, 7]]] ]

Out[15]= True
```

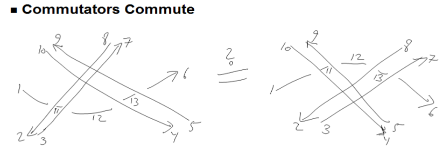


```
In[10]:= Timing[res4 = pA[#, (t[1] == t[6]) (t[11] == t[16]) (t[21] == t[26])] & /@ {
           CircuitDiagram[
            Xm[19, 1, 20, 2], Xp[11, 3, 12, 2], Xp[3, 30, 4, 29], Xm[4, 21, 5, 22],
            Xp[6, 23, 7, 22], Xm[7, 28, 8, 29], Xm[12, 8, 13, 9], Xp[18, 10, 19, 9],
            Xm[27, 13, 28, 14], Xp[23, 15, 24, 14], Xm[24, 16, 25, 17], Xp[26, 18, 27, 17]
           ],
           CircuitDiagram[
            Xp[1, 28, 2, 27], Xm[2, 23, 3, 24], Xm[17, 3, 18, 4], Xp[13, 5, 14, 4],
            Xm[14, 6, 15, 7], Xp[16, 8, 17, 7], Xp[8, 25, 9, 24], Xm[9, 26, 10, 27],
            Xm[29, 11, 30, 12], Xp[21, 13, 22, 12], Xm[22, 18, 23, 19], Xp[28, 20, 29, 19]
           ]
          }]

         A very large output was generated. Here is a sample of it:
Out[10]= {9.86, {AHD[(t[1] == t[2] == t[3] == t[4] == t[5] == t[6] == t[7] == t[8] == t[9] == t[10])
             (t[11] == t[12] == t[13] == t[14] == t[15] == t[16] == t[17] == t[18] == t[19] == t[20])
             (t[21] == t[22] == t[23] == t[24] == t[25] == t[26] == t[27] == t[28] == t[29] == t[30]),
           {1, 6, 11, 16, 21, 26}, <<1>>,
           -t[1]^2 t[11]^2 t[21]^2 W[1, 5, 6, 11, 15, 21] + <<2574>>], <<1>>}}
         Show Less | Show More | Show Full Output | Set Size Limit...

In[11]:= Equal @@ (Last /@ res4)

Out[11]= True
```

(The program also prints "False" when appropriate, and computes Alexander polynomials)

More at http://www.math.toronto.edu/~drorbn/Sandbjerg-0810/pA.nb

▪ Commutators Commute



```
In[5]:= Equal[
         pA[CircuitDiagram[Xp[1, 2, 11, 8], Xm[11, 3, 12, 7],
           Xp[12, 4, 13, 10], Xm[13, 5, 6, 9]], t[2] == t[3], t[4] == t[5]],
         pA[CircuitDiagram[Xp[1, 4, 11, 10], Xm[11, 5, 12, 9],
           Xp[12, 2, 13, 8], Xm[13, 3, 6, 7]], t[2] == t[3], t[4] == t[5]] ]

Out[5]= True
```

Question.

Does this specify the Alexander polynomial?

References

[Ar] J. Archibald, *The Weight System of the Multivariable Alexander Polynomial,* arXiv:0710.4885.

[MH] H. Murakami, *A Weight System Derived from the Multivariable Conway Potential Function,* Jour. of the London Math. Soc. **59** (1999) 698–714, arXiv:math/9903108.

[MJ] J. Murakami, *A State Model for the Multi-Variable Alexander Polynoomial,* Pac. Jour. of Math. **157-1** (1993) 109–135.

[NS] S. Naik and T. Stanford, *A Move on Diagrams that Generates S-Equivalence of Knots,* Jour. of Knot Theory and its Ramifications **12-5** (2003) 717–724, arXiv:math/9911005.

[Va] A. Vaintrob, *Melvin-Morton Conjecture and Primitive Feynman Diagrams,* Inter. J. Math. **8** (1997) 537–553, arXiv:q-alg/9605028.

More at http://www.math.toronto.edu/~drorbn/Talks/Sandbjerg-0810/