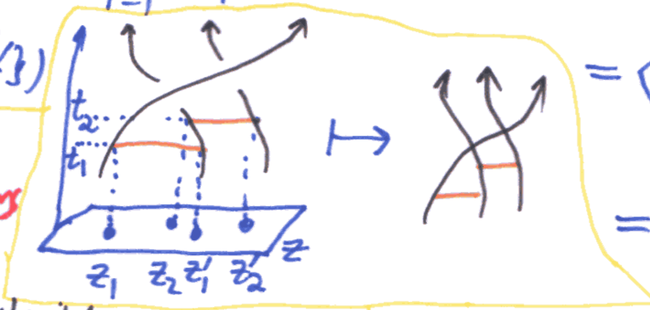
Math 1352 Algebraic Knot Theory – The Knizhnik-Zamolodchikov Connection

Theorem 1. The following is an invariant of braids in $\mathbb{R}_t\times\mathbb{C}_z$ (Fixed endpoints)

$$Z(B)=\oint_{\substack{t_1\le\dots\le t_m\\ P=(\{z_i,z_i'\})}}\frac{D_P}{(2\pi i)^m}\bigwedge_{i=1}^{m}\frac{dz_i-dz_i'}{z_i-z_i'}\quad\text{in}\quad \mathcal{A}(\uparrow_n):=\left\langle t^{ij}:1\le i\ne j\le n\right\rangle\Big/\begin{matrix}t^{ij}=t^{ji},\\ [t^{ij},t^{kl}]=0\\ [t^{ij},t^{ik}+t^{jk}]=0\end{matrix}$$



$=\Big\langle$  $\Big\rangle\Big/4T$

= "Chord diagrams for braids".

Formal Connections & Curvature.

Let $\Omega\in\Omega^1(M,A)$ (graded unital.) with $\deg\Omega=1$.
$\gamma:[0,1]=I\to M$ induces
$\phi:\Delta^m=\{0\le t_1\le\dots\le t_m\le 1\}\to M^m$.
Set $hol_\gamma(\Omega)=P\exp\int_\gamma\Omega=\oint_{\Delta^m}\phi^*\Omega^m$
where $\Omega^m:=\pi_1^*\Omega\wedge\dots\wedge\pi_m^*\Omega$

Theorem 2. If $F_\Omega:=d\Omega+\Omega\wedge\Omega=0$, then $hol(\Omega)$ is invariant under end-point preserving homotopies of γ.

Proof 2. Let $\Gamma:I_s\times I_t\to M$, $\Phi:I_s\times\Delta^m\to M^m$; By Stokes',

$$\int_{\Delta^m}\Phi_1^*\Omega^m-\int_{\Delta^m}\Phi_0^*\Omega^m=\int_{I\times\Delta^m}d\Phi^*\Omega^m-\int_{I\times\partial\Delta^m}\Phi^*\Omega^m=:A_m-B_m$$

Now
$$A_m=\sum_{k=1}^{m}(-1)^{k+1}\int_{I\times\Delta^m}\pi_1^*\Omega\wedge\dots\wedge\pi_k^*d\Omega\wedge\dots\wedge\pi_m^*\Omega$$
and (Fixed endpoints)
$$B_m=\overset{0}{\int_{I\times[t_1=0]}\Phi^*\Omega^m}\pm\overset{0}{\int_{I\times[t_m=1]}\Phi^*\Omega^m}+\sum_{k=1}^{m-1}(-1)^k\int_{I\times[t_k=t_{k+1}]}\Phi^*\Omega^m$$
$$=\sum_{k=1}^{m-1}(-1)^k\int_{I\times\Delta^{m-1}}\pi_1^*\Omega\wedge\dots\wedge\pi_k^*(\Omega\wedge\Omega)\wedge\dots\wedge\pi_{m-1}^*\Omega$$

and now $\sum A_m=\sum B_m$ by telescopic summation & $F_\Omega=0$.

The KZ connection.

$M=\mathbb{C}^n\setminus\{\text{diagonals}\}$, $A=\mathcal{A}(\uparrow_n)$,
and $\Omega=\sum_{i<j}t^{ij}w_{ij}$ where $w_{ij}:=\dfrac{dz_i-dz_j}{z_i-z_j}\underset{\text{locally}}{=}d\log(z_i-z_j)$

Compute $F_\Omega=d\Omega+\Omega\wedge\Omega$: $dw_{ij}=0$ so $d\Omega=0$.

$$\Omega\wedge\Omega=\sum_{\substack{i<j\\k<l}}t^{ij}t^{kl}w_{ij}\wedge w_{kl}=\underbrace{A}_{|\{ijkl\}|=2}+\underbrace{B}_{=3}+\underbrace{C}_{=4}\quad\text{where}$$

$A=C=0$ as $[t^{ij},t^{kl}]=0$ if $|\{ijkl\}|=2$ or 4 and

$$B=\sum_{\alpha<\beta<\gamma}[t^{\alpha\beta},t^{\beta\gamma}]w_{\alpha\beta}\wedge w_{\beta\gamma}+\text{Cyclic perms}$$
$$=\sum_{\alpha<\beta<\gamma}Y^{\alpha\beta\gamma}\big(w_{\alpha\beta}\wedge w_{\beta\gamma}+\text{cyc perms}\big)=0$$

=0 by "Arnold's identity"

Note: by 4T, $[t^{\alpha\beta},t^{\beta\gamma}]=Y^{\alpha\beta\gamma}=$ 

Proof of 1 Simply take in theorem 2, $\gamma=$ the braid and $\Omega=$ the KZ connection.

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