



M. Khovanov

Local Differentials and Matrix Factorizations

L. Rozansky



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Quantum algebra:

where

Claim. If $ba=qab$ then

$$(n)_q := 1 + q + \dots + q^{n-1},$$

$$(n)!_q := (1)_q(2)_q \cdots (n)_q,$$

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k}_q a^k b^{n-k}$$

$$\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q(n-k)!_q}.$$

Conjecture:

(I. Frenkel, though he may disown this version)

1. Every object in mathematics is the Euler characteristic of a complex.
2. Every operation in mathematics lifts to an operation between complexes.
3. Every identity in mathematics is true up to homotopy at complex-level.

I. Frenkel



Local state spaces:

$$V = \left\langle \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \end{array} \right\rangle$$

$$V^{\otimes(4 \times 5)} = \left\langle \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \end{array} ; \dots \right\rangle$$

Local differentials:

$$d \left(\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline d & \\ \hline & \\ \hline \end{array} - \begin{array}{|c|c|} \hline & d \\ \hline & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline d & \\ \hline \end{array} - \begin{array}{|c|c|} \hline & \\ \hline & d \\ \hline \end{array}$$

where

$$d^2 \left(\begin{array}{|c|c|} \hline \text{diagram 1} \\ \hline \end{array} \right) = 0 \text{ or } d^2 \left(\begin{array}{|c|c|} \hline \text{diagram 2} \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \text{diagram 3} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{diagram 4} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{diagram 5} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{diagram 6} \\ \hline \end{array}$$

Tagged doodles:

(degrees in orange)

$$\begin{array}{c} \text{diagram 1} \rightarrow \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \\ \text{diagram 2} \rightarrow \left\{ \begin{array}{c} -1 \\ 00 \end{array} \right\}, \left\{ \begin{array}{c} 1-n \\ 11 \end{array} \right\}, \left\{ \begin{array}{c} 3-2n \\ 10 \end{array} \right\}, \left\{ \begin{array}{c} -n \\ 01 \end{array} \right\} \\ \text{diagram 3} \rightarrow \left\{ \begin{array}{c} x \\ x \end{array} \right\} := x^3 \end{array}$$

$$d \left(\begin{array}{|c|} \hline x \\ \hline \end{array} \right) := \begin{array}{|c|} \hline x \\ \hline \end{array} - \begin{array}{|c|} \hline 0 \\ \hline \end{array} = (x-y) \quad d \left(\begin{array}{|c|} \hline x \\ \hline \end{array} \right) := \pi \begin{array}{|c|} \hline x \\ \hline \end{array}$$

$$\text{In}[1]:= n=2; \pi_{i,j} := \text{Cancel} \left[\frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j} \right]; \pi_{1,2}$$

$$\text{Out}[1]= x_1^2 + x_1 x_2 + x_2^2$$

$$\text{In}[2]:= L = \begin{pmatrix} 0 & x_1 - x_2 \\ \pi_{1,2} & 0 \end{pmatrix};$$

Expand[L.L] // MatrixForm

(deg d = n+1)

$$\text{Out}[3]/\text{MatrixForm} = \begin{pmatrix} x_1^3 - x_2^3 & 0 \\ 0 & x_1^3 - x_2^3 \end{pmatrix}$$

Matrix factorizations:

$$\begin{array}{ccccc} M^0 & \xrightarrow{A} & M^1 & \xrightarrow{B} & M^0 \\ U^0 \downarrow V^0 & & U^1 \downarrow V^1 & & U^0 \downarrow V^0 \\ N^0 & \xrightarrow{A'} & N^1 & \xrightarrow{B'} & N^0 \end{array}$$

A category, with "complexes", morphisms, homotopies, direct sums and tensor products.

D. Eisenbud



See Khovanov and Rozansky, arXiv:math.QA/0401268

Likewise, set $Q=d$ with:

$$\text{In}[4]:= Q := \begin{pmatrix} 0 & 0 & v_1 & v_2 \\ 0 & 0 & u_2 & -u_1 \\ u_1 & v_2 & 0 & 0 \\ u_2 & -v_1 & 0 & 0 \end{pmatrix};$$

$$\{v_1, v_2\} = \{x_1 + x_2 - x_3 - x_4, x_1 x_2 - x_3 x_4\};$$

$$\text{In}[6]:= g[s_, p_] :=$$

$$s^{n+1} + (n+1) \sum_{i=1}^{(n+1)/2} \frac{(-1)^i}{i} \text{Binomial}[n-i, i-1] s^{n+1-2i} p^i;$$

g[x+y, x y] // Expand

$$\text{Out}[6]= x^3 + y^3$$

$$\text{In}[7]:= \{u_1, u_2\} =$$

$$\text{Cancel} \left[\left\{ \frac{g[x_1 + x_2, x_1 x_2] - g[x_3 + x_4, x_1 x_2]}{v_1}, \frac{g[x_3 + x_4, x_1 x_2] - g[x_3 + x_4, x_3 x_4]}{v_2} \right\} \right]$$

$$\text{Out}[7]= \{x_1^2 - x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2 + x_1 x_4 + x_2 x_4 + 2 x_3 x_4 + x_4^2, -3(x_3 + x_4)\}$$

$$\text{In}[8]:= \omega = u_1 v_1 + u_2 v_2 // \text{Expand}$$

$$\text{Out}[8]= x_1^3 + x_2^3 - x_3^3 - x_4^3$$

$$\text{In}[9]:= \text{Simplify}[Q.Q == \omega \text{IdentityMatrix}[4]]$$

$$\text{Out}[9]= \text{True}$$

Example: Set $P=d$

$$P = \begin{pmatrix} 0 & 0 & x_1 - x_4 & x_2 - x_3 \\ 0 & 0 & \pi_{2,3} & -\pi_{1,4} \\ \pi_{1,4} & x_2 - x_3 & 0 & 0 \\ \pi_{2,3} & x_4 - x_1 & 0 & 0 \end{pmatrix};$$

$$\text{In}[11]:= \text{Simplify}[P.P == \omega \text{IdentityMatrix}[4]]$$

$$\text{Out}[11]= \text{True}$$

Theorem: (Kh-Ro) Taking homology and then the graded Euler characteristics, we get the [MOY] relations:

$$\begin{array}{c} \uparrow \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array} \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} = [2] \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} = [n-1]$$

$$\begin{array}{c} \uparrow \\ \downarrow \end{array} \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array} + [n-2] \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} \quad \begin{array}{c} \uparrow \\ \downarrow \end{array}$$

$$\begin{array}{c} \uparrow \\ \downarrow \end{array} \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} + \begin{array}{c} \uparrow \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array} \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} + \begin{array}{c} \uparrow \\ \downarrow \end{array}$$

[MOY] := Murakami, Ohtsuki, Yamada, Enseignement Math. 44 (1998)

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}}$$

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