
# From Stonehenge to Witten – Some Further Details

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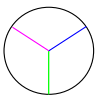
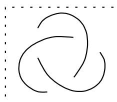
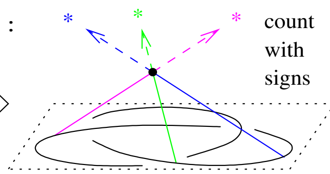
We the generating function of all stellar coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\overline{\Pi}} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

$$\langle D, K\rangle_{\overline{\Pi}} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix} :$$

$D=$  $K=$  $\Longrightarrow$ 

count with signs

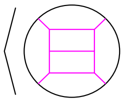
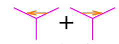
**Theorem.** Modulo Relations, *Z(K)* is a knot invariant!

Dylan Thurston

$N$ := # of stars

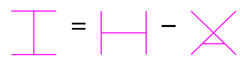
$c$ := # of chopsticks

$e$ := # of edges of $D$

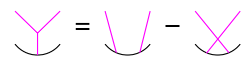
$\mathcal{A}(\circlearrowleft)$ :=Span $\left\langle\right.$  $\left.\right\rangle$ / oriented vertices, AS:  =0 & more relations

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,

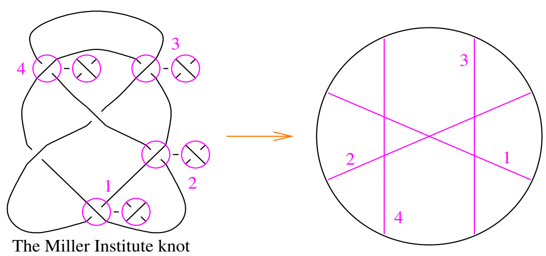


An intersection line cuts through the knot – Solution: Impose STU,



The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right] \longrightarrow \sum_{\substack{D:\text{ Feynman}\\ \text{diagram}}} W_{\mathfrak{g}}(D) \not\sum \mathcal{E}(D) \longrightarrow \sum_{\substack{D:\text{ Feynman}\\ \text{diagram}}} D \not\sum \mathcal{E}(D)$$




The Miller Institute knot

**Definition.** $V$ is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

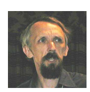
**Theorem.** All knot polynomials (Conway, Jones, etc.) are of finite type.

**Conjecture.** (Taylor's theorem) Finite type invariants separate knots.

**Theorem.** *Z(K)* is a universal finite type invariant!

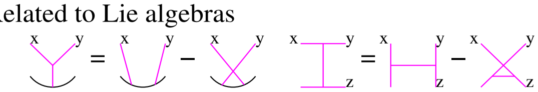
(sketch: to dance in many parties, you need many feet).

Goussarov

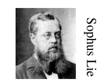


Vassiliev

Related to Lie algebras



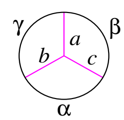

[x,y] = xy − yx    [[x,y],z]=[x,[y,z]]−[y,[x,z]]




More precisely, let $\mathfrak{g} = \langle X_a\rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha\rangle$ be a representation. Set

$$f_{abc} := \langle [a,b], c\rangle \qquad X_a v_\beta = \sum_\beta r^\beta_{\alpha\gamma} v_\gamma$$

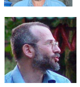
and then

$W_{\mathfrak{g},R}$ :  $\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$



$W_{\mathfrak{g},R} \circ Z$ is often interesting:

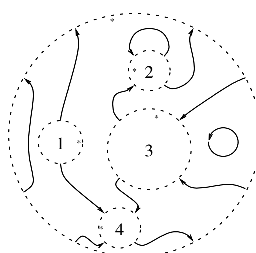
$\mathfrak{g} = sl(2)$ $\longrightarrow$  The Jones polynomial

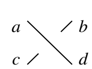
$\mathfrak{g} = sl(N)$ $\longrightarrow$  The HOMFLYPT polynomial

Przytycki

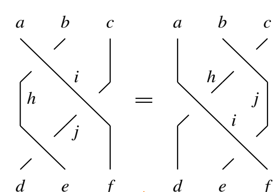
$\mathfrak{g} = so(N)$ $\longrightarrow$  The Kauffman polynomial

Planar algebra and the Yang–Baxter equation



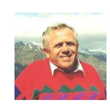




$R^{ab}_{cd}$

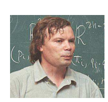



$$R^{ab}_{hi} R^{ic}_{jf} R^{hj}_{de} = R^{ah}_{di} R^{bc}_{hj} R^{ij}_{ef}$$

Yang

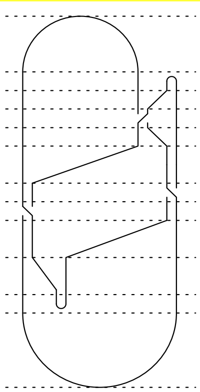


Baxter

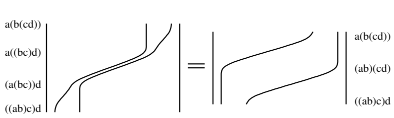


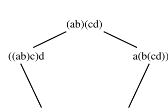
Reshetikhin

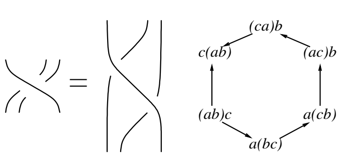
Turaev



Parenthesized tangles, the pentagon and hexagon




Drinfel'd

Kauffman's bracket and the Jones polynomial

$\langle \times \rangle = \langle \asymp \rangle - q\langle )( \rangle$ 0-smoothing 1-smoothing

$\langle \bigcirc^k \rangle = (q+q^{-1})^k$

$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$

$(n_+, n_-)$ count $(\nearrow\!\!\!\nwarrow, \nwarrow\!\!\!\nearrow)$

Claim $\hat{J}(\,)\!\!(\,) = \hat{J}(\,)(\,)$

Indeed,

$\langle \ \rangle = \langle \ \rangle - q\langle \ \rangle$

$-q\langle \ \rangle + q^2\langle \ \rangle$

$(q+q^{-1})$

$= -q\langle )( \rangle$

"God created the knots, all else in topology is the work of man."    This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407

More at http://www.math.toronto.edu/~drorbn/Talks/Oporto-0407/