



M. Khovanov

Local Differentials and Matrix Factorizations

Dror Bar-Natan at UIUC, March 11, 2004, <http://www.math.toronto.edu/~drorbn/Talks/UIUC-050311/>

L. Rozansky

**Quantum algebra:**Claim. If $ba=qab$ then

where

$$(n)_q := 1 + q + \dots + q^{n-1},$$

$$(n)!_q := (1)_q (2)_q \cdots (n)_q,$$

$$\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q (n-k)!_q}.$$

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k}_q a^k b^{n-k}$$

Conjecture:

(I. Frenkel, though he may disown this version)

1. Every object in mathematics is the Euler characteristic of a complex.
2. Every operation in mathematics lifts to an operation between complexes.
3. Every identity in mathematics is true up to homotopy at complex-level.

I. Frenkel

**Local state spaces:**

$$V = \left\langle \begin{array}{c} \text{doodle} \\ \text{doodle} \\ \text{doodle} \\ \text{doodle} \\ \text{doodle} \\ \text{doodle} \end{array} \right\rangle$$

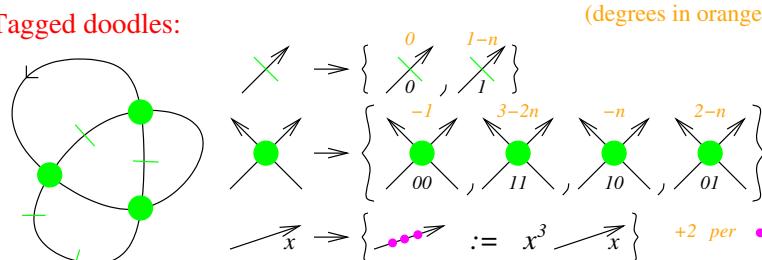
$$V^{\otimes (4 \times 5)} = \left\langle \begin{array}{c} \text{doodle} \\ \text{doodle} \\ \text{doodle} \\ \text{doodle} \\ \text{doodle} \end{array} ; \begin{array}{c} \text{doodle} \\ \text{doodle} \\ \text{doodle} \\ \text{doodle} \\ \text{doodle} \end{array} ; \dots \right\rangle$$

Local differentials:

$$d \left(\begin{array}{|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) = + \begin{array}{|c|c|} \hline d & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & d \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & & \\ \hline & d & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & & \\ \hline & & d \\ \hline & & \\ \hline \end{array}$$

where

$$d^2 \left(\begin{array}{|c|c|} \hline \text{smiley} & \text{smiley} \\ \hline \text{smiley} & \text{smiley} \\ \hline \end{array} \right) = 0 \quad \text{or} \quad d^2 \left(\begin{array}{|c|c|} \hline \text{smiley} & \text{smiley} \\ \hline \text{smiley} & \text{smiley} \\ \hline \end{array} \right) = + \begin{array}{|c|c|} \hline \text{smiley} & \text{smiley} \\ \hline \text{smiley} & \text{smiley} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{smiley} & \text{smiley} \\ \hline \text{smiley} & \text{smiley} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{smiley} & \text{smiley} \\ \hline \text{smiley} & \text{smiley} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{smiley} & \text{smiley} \\ \hline \text{smiley} & \text{smiley} \\ \hline \end{array}$$

Tagged doodles:

$$d \nearrow \begin{array}{c} 1 \\ 0 \end{array} := \nearrow \begin{array}{c} 0 \\ 1 \end{array} - \nearrow \begin{array}{c} 0 \\ 0 \end{array} = (x-y) \quad d \nearrow \begin{array}{c} x \\ 0 \end{array} := \pi \nearrow \begin{array}{c} 1 \\ y \end{array}$$

$$\text{In[1]:= } n = 2; \pi_{i,j} := \text{Cancel}\left[\frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j}\right]; \pi_{1,2}$$

$$\text{Out[1]= } x_1^2 + x_1 x_2 + x_2^2$$

$$\text{In[2]:= } L = \begin{pmatrix} 0 & x_1 - x_2 \\ \pi_{1,2} & 0 \end{pmatrix}; \text{Expand}[L.L] // \text{MatrixForm}$$

$$\left\{ \text{Set } L=d \right| \nearrow \quad (\deg d = n+1)$$

Out[3]/MatrixForm=

$$\begin{pmatrix} x_1^3 - x_2^3 & 0 \\ 0 & x_1^3 - x_2^3 \end{pmatrix}$$

Matrix factorizations:

$$D = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \quad \begin{array}{ccc} M^0 & \xrightarrow{A} & M^1 \\ U^0 \downarrow V^0 & \swarrow h^1 & \downarrow U^1 V^1 \\ N^0 & \xrightarrow{A'} & N^1 \end{array} \quad \begin{array}{ccc} M^1 & \xrightarrow{B} & M^0 \\ U^1 \downarrow V^1 & \swarrow h^0 & \downarrow U^0 V^0 \\ N^1 & \xrightarrow{B'} & N^0 \end{array}$$

AB = BA = ωI
A category, with "complexes", morphisms, homotopies, direct sums and tensor products.

D. Eisenbud

See Khovanov and Rozansky, arXiv:math.QA/0401268

Likewise, set $Q=d$ | with:

$$\text{In[4]:= } Q := \begin{pmatrix} 0 & 0 & v_1 & v_2 \\ 0 & 0 & u_2 & -u_1 \\ u_1 & v_2 & 0 & 0 \\ u_2 & -v_1 & 0 & 0 \end{pmatrix};$$

$$\{v_1, v_2\} = \{x_1 + x_2 - x_3 - x_4, x_1 x_2 - x_3 x_4\};$$

$$\text{In[6]:= } g[s_, p_] := s^{n+1} + (n+1) \sum_{i=1}^{(n+1)/2} \frac{(-1)^i}{i} \text{Binomial}[n-i, i-1] s^{n+1-2i} p^i;$$

$$g[x+y, x y] // \text{Expand}$$

$$\text{Out[6]= } x^3 + y^3$$

$$\text{In[7]:= } \{u_1, u_2\} = \text{Cancel}\left[\left\{ \frac{g[x_1 + x_2, x_1 x_2] - g[x_3 + x_4, x_1 x_2]}{v_1}, \frac{g[x_3 + x_4, x_1 x_2] - g[x_3 + x_4, x_3 x_4]}{v_2} \right\} \right]$$

$$\text{Out[7]= } \{x_1^2 - x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2 + x_1 x_4 + x_2 x_4 + 2 x_3 x_4 + x_4^2, -3 (x_3 + x_4)\}$$

$$\text{In[8]:= } \omega = u_1 v_1 + u_2 v_2 // \text{Expand}$$

$$\text{Out[8]= } x_1^3 + x_2^3 - x_3^3 - x_4^3$$

$$\text{In[9]:= } \text{Simplify}[Q.Q == \omega \text{IdentityMatrix}[4]]$$

$$\text{Out[9]= True}$$

$$\text{Example: Set } P=d \quad | \quad \nearrow \quad \text{In[10]:= } P = \begin{pmatrix} 0 & 0 & x_1 - x_4 & x_2 - x_3 \\ 0 & 0 & \pi_{2,3} & -\pi_{1,4} \\ \pi_{1,4} & x_2 - x_3 & 0 & 0 \\ \pi_{2,3} & x_4 - x_1 & 0 & 0 \end{pmatrix};$$

$$\begin{array}{c} 1 \\ \nearrow \\ 4 \\ \searrow \\ 3 \\ 2 \end{array}$$

$$\text{In[11]:= } \text{Simplify}[P.P == \omega \text{IdentityMatrix}[4]]$$

$$\text{Out[11]= True}$$

Theorem: (Kh-Ro) Taking homology and then the graded Euler characteristics, we get the [MOY] relations:

$$\begin{array}{c} \uparrow \\ \nearrow \\ \text{Diagram} \\ \searrow \\ \uparrow \end{array} = [2] \quad \begin{array}{c} \nearrow \\ \text{Diagram} \\ \searrow \\ \nearrow \end{array} = [n-1] \quad \begin{array}{c} \nearrow \\ \text{Diagram} \\ \searrow \\ \nearrow \end{array} = [n-2] \quad \begin{array}{c} \nearrow \\ \text{Diagram} \\ \searrow \\ \nearrow \end{array} + \begin{array}{c} \nearrow \\ \text{Diagram} \\ \searrow \\ \nearrow \end{array} \quad \begin{array}{c} \nearrow \\ \text{Diagram} \\ \searrow \\ \nearrow \end{array} + \begin{array}{c} \nearrow \\ \text{Diagram} \\ \searrow \\ \nearrow \end{array} = \begin{array}{c} \nearrow \\ \text{Diagram} \\ \searrow \\ \nearrow \end{array} + \begin{array}{c} \nearrow \\ \text{Diagram} \\ \searrow \\ \nearrow \end{array}$$

[MOY] := Murakami, Ohtsuki, Yamada,
Enseignement Math. 44 (1998)

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}}$$