FUNDAMENTAL CONCEPTS IN DIFFERENTIAL GEOMETRY
FALL 2000
EXERCISES HANDOUT # 10

1. Show that $\wedge$ is a commutative and associative operation on $A^*(V)$, where V is a finite dimensional vector space.

2. Show that the two definitions given in class for the smoothness of a differential form $\omega \in \Omega^p(M)$ are equivalent.

3. *(The Hodge operator.)* Let $V \cong \mathbb{R}^n$. Suppose that V is equipped with the additional data of:

(i) A non degenerate symmetric bilinear form $B: V \times V \to \mathbb{R}$.
(ii) A nonzero top form $\nu \in A^n(V)$.

It is a standard fact that B induces an isomorphism $V \cong V^*$, hence V^* is also naturally equipped with a non degenerate symmetric bilinear form also denoted B, for obvious reasons.

(a) Show that B induces a symmetric bilinear form $B^{\otimes k}$ on $V^{*\otimes k}$ which is unique under the requirement that for 1-forms $\omega_1, \dots, \omega_k$ and $\eta_1, \dots, \eta_k$

$$B^{\otimes k}(\omega_1 \otimes \cdots \otimes \omega_k, \eta_1 \otimes \cdots \otimes \eta_k) = \prod_{i=1}^{k} B(\omega_i, \eta_i).$$

Recall that $A^k(V)$ is a subspace of $V^{*\otimes k}$, hence $B^{\otimes k}$ induces a bilinear form on $A^k(V)$ by

$$B^{\wedge k} = \frac{1}{k!} B^{\otimes k}|_{A^k(V)}$$

Show that $B^{\wedge k}$ is non degenerate (and symmetric). Show further, that if $x_1, \dots, x_n$ is a basis for V, then $B^{\wedge n}(dx_1 \wedge \ldots \wedge dx_n, dx_1 \wedge \ldots \wedge dx_n) = \det\{B(dx_i, dx_j)\}$. For this reason, we usually assume that $B^{\wedge n}(\nu, \nu) = \pm 1$.

(b) Use ν to canonically identify $A^n(V)$ with $\mathbb{R}$. Then, every $\alpha \in A^{n-k}(V)$ induces a linear functional $A^k(V) \to \mathbb{R}$ via the assignment $\eta^p \mapsto \eta^p \wedge \alpha$. Show that this induces a canonical isomorphism $A^{n-k}(V) \cong A^k(V)^*$.

(c) Deduce that for every $\alpha \in A^k(V)$ there exists a unique $(n-k)$-form $\star\alpha$ satisfying the following equation for every k-form η:

$$\eta \wedge \star\alpha = B^{\wedge k}(\eta, \alpha)\nu.$$

Show that $\star$ is a linear isomorphism $A^k(V) \cong A^{n-k}(V)$. Show also that if one chooses ν as in the end of (a), then, up to sign

$$B^{\wedge k}(\alpha, \beta) = B^{\wedge(n-k)}(\star\alpha, \star\beta).$$

(d) Let $V = \mathbb{R}^3$ be equipped with the usual inner product and top form. Let α, β be 1-forms and compute $\star(\alpha \wedge \beta)$. Is this result familiar ?

(e) Let $V = \mathbb{R}^4$ be equipped with the Lorenz form $B(x, y) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$ and the obvious top form. Compute $\star$.

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