FUNDAMENTAL CONCEPTS IN DIFFERENTIAL GEOMETRY
FALL 2000
EXERCISES HANDOUT # 13

Definition. A k-form $\omega \in \Omega^k(M)$ is called *closed* if $d\omega = 0$. It is called *exact* if $\omega = d\eta$ for some $(k-1)$-form η.

1. Let $f : \mathbb{R} \to S^1$ be given by $f(t) = e^{it}$.

(a) Show that there exists a unique $\omega \in \Omega^1(S^1)$ such that $f^*(\omega) = dt$.
(b) Show directly from the definition that ω is closed.
(c) Show that there exists no $g \in \Omega^0(S^1)$ such that $dg = \omega$.

2. Show, directly from the definition, that every closed 1-form on $\mathbb{R}^2$, is exact.

3. Compute $\int_{S^2} z dx \wedge dy$ using Stokes' theorem. Then compute the integral directly by parametrizing the Northern and Southern hemispheres and pulling back the form. Expain how you determine whether to add or substract the two results in order to obtain the correct figure.

4. Consider the following 2-form in $\mathbb{R}^3$, $\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$.

(a) Let u, v be vectors in $T_p\mathbb{R}^3$. Show that $\omega_p(u, v) = \langle p, u \times v \rangle$.
(b) Write ω in spherical coordinates r, θ, φ.
(c) Show that ω is preserved under rotations about the z-axis, and an apropriate "switch" of coordinates. Beforehand, explain rigorously what you are proving.
(d) Conclude that ω is preserved under $SO(3)$ either by using (a) or (b)+(c).

5. (a) Consider the following 2-form in $\mathbb{R}^3 - \{0\}$,

$$\omega = (x^2 + y^2 + z^2)^\alpha (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy).$$

Find a real number α such that ω is closed. It is advisable to use 4(b).

(b) Let S^2 denote the unit sphere in $\mathbb{R}^3$. Compute $\int_{S^2} \omega$.
(c) Is ω exact in $\mathbb{R}^3$? In the half space $x > 0$?
(d) Compute $\int_{S^2(a,b,c,r)} \omega$, where $S^2(a, b, c, r)$ is a sphere of radius r about (a, b, c) in $\mathbb{R}^3$.
(e) Questions (a)-(d) can be formulated for a 1-form in $\mathbb{R}^2 - \{0\}$

$$\omega = (x^2 + y^2)^\alpha (x dy - y dx).$$

Answer (a)-(d) in this case if you could not handle the 3 dimensional case.

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