Some Hints for Solutions

1 Exercise #1 in handout #2

(1) Suppose that $\varphi : \mathbb{R} \to X$ is a diffeomorphism, and let $\psi : X \to \mathbb{R}$ be its inverse. Suppose that $\varphi(0) = 0$.

Let $j : X \to \mathbb{R}^2$ be the inclusion map. Then j is smooth (why ?).

By the definition of F_X, there exists a smooth map Ψ extending ψ and rendering the following diagram commutative

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\varphi"] \arrow[dr, "\Phi"'] & X \arrow[r, "\psi"] \arrow[d, "j"] & \mathbb{R} \\
& \mathbb{R} \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

(2) $\Psi\Phi = \Psi j \varphi = \psi\varphi = \mathrm{id}$ implies d
$Phi_0 \neq 0$, namely $\Phi'(0) \neq 0$.

(3) Connectedness consideration of $X - \{0\}$ and $\mathbb{R} - \{0\}$ implies that we may assume $\Phi((-\infty, 0)) \subseteq (-\infty, 0) \times \mathbb{R}$ and $\Phi((0, \infty)) \subseteq (0, \infty) \times \mathbb{R}$. Therefore (explain this)

$$\Phi'(0) = \lim_{t \uparrow 0} \frac{\Phi(t) - \Phi(0)}{t} = (a, -a)$$
$$\Phi'(0) = \lim_{t \downarrow 0} \frac{\Phi(t) - \Phi(0)}{t} = (b, b).$$

It follows that $a = b = 0$ hence $\Phi'(0) = 0$, a contradiction.

2 Exercise #2 in handout #3

The following argument is due to Yariv. Use paths to represent derivations. The inclusion $j : U \to M$ induces $j_* : T_pU \to T_pM$ via $j_*(D_\gamma) = D_{j\gamma}$.

Define $\pi : T_pM \to T_pU$ as follows. For $\gamma : (-a, a) \to M$ there exists ϵ such that $\gamma((-\epsilon, \epsilon)) \subset U$. Define

$$\pi(D_\gamma) = D_{\gamma|_{(-\epsilon,\epsilon)}}.$$

This is independent in the choice of ϵ and of γ (why ?). It is almost trivial that j_* and π are inverses of one another.

Another easy proof can be obtained using the definition of the tangent space as in exercise #1 in this handout. Surjectivity of j_* is obvious and dimensional argument concludes.

What I said in class was rubbish. It will make sense when we talk about derivations $C^\infty(M) \to C^\infty(M)$ which correspond to vector fields. The best is yet to come.

3 Exercise #3 in handout #3

Identify $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ via

$$\frac{\partial}{\partial x_i} \mapsto e_i.$$

Under this identification, if a path $\gamma\colon(-a,a) \to \mathbb{R}^n$ represents a tangent vector in $T_p\mathbb{R}^n$, then γ corresponds to $\gamma'(0)$ in $\mathbb{R}^n$. To see this recall that using the trivial chart on $\mathbb{R}^n$,

$$D_\gamma = \sum_{i=1}^n \frac{d\gamma^i}{dt}(0)\frac{\partial}{\partial x_i}.$$

In our example

$$\Psi_*(\frac{\partial}{\partial\theta}) \equiv D_{(\sin\varphi\cos(\theta+t),\sin\varphi\sin(\theta+t),\cos\varphi)\subset S^2} \stackrel{j_*}{\to}$$
$$D_{(\sin\varphi\cos(\theta+t),\sin\varphi\sin(\theta+t),\cos\varphi)\subset\mathbb{R}^3} \sim (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$$

A similar computation shows that

$$\Psi_*(\frac{\partial}{\partial\varphi}) \sim (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi).$$

One checks directly that these are nonzero orthogonal vectors and both are orthogonal to the vector $\Psi(\theta,\varphi)$.

This argument also shows that the inclusion map $i\colon S^2 \to \mathbb{R}^3$ is an immersion because S^2 is a 2-dimensional manifold and i_* has a 2 dimensional space as its image. Of course, you may (and better) use the theorem proved in class about the inverse image of a regular value.