ALGEBRAIC TOPOLOGY, PROBLEM SET 6

1. Calculation of the fundamental group of $SO(3)$

$SO(3)$ is the group of automorphisms of the 3-dimensional Euclidian space which preserve an orientation. The goal of the current set of exercises is to understand $\pi_1(SO(3), I)$, where I is the identity matrix.

Exercise 1.1. Let E be the set of all traceless, anti-hermitian 2×2-matrices, i.e.

$$E = \{A \in M_2(\mathbb{C}) \mid \mathrm{Tr}(A) = 0; \bar{A}^t = -A\}.$$

Introduce on E a structure of the 3-dimensional Euclidian space. (*Hint:* No tricks ! Use the most natural structure you can think of.)

Exercise 1.2. $SU(2)$ is the set of all complex 2×2-matrices with determinant 1 which preserve 'one and a half'-linear form on $\mathbb{C}^2$. Equivalently,

$$SU(2) = \{U \in M_2(\mathbb{C}) \mid \bar{U}^t U = I, \det(U) = 1\}$$

Find a homeomorphism $SU(2) \cong S^3$.

Exercise 1.3. Find the action of $SU(2)$ on E which preserves the Euclidian structure and orientation. (*Hint:* Adjunction will work.)

Exercise 1.4. Explain why in the previous exercise we've constructed a homomorphism $p\colon SU(2) \to SO(3)$.

Exercise 1.5. Consider $\ker(p)$ from the previous exercise and conclude that p is a (universal) covering map.

Exercise 1.6. Find $\pi_1(SO(3))$.

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