Topological Theorems About $\mathbb{R}^n$

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Theorem 1. *There is no continuous retract* $S^{n-1} \to D^n$.

Theorem 2. *(The Brouwer fixed point theorem, ∼1910) Every continuous map* $f : D^n \to D^n$ *has a fixed point.*

Theorem 3. *If* $n \neq m$ *then* S^n *is not homeomorphic to* S^m *and* $\mathbb{R}^n$ *is not homeomorphic to* $\mathbb{R}^m$.

Theorem 4. *There are no continuous non-vanishing vector fields on even-dimensional spheres.*

Theorem 5. *If* D *is an embedded closed disk in* S^n *then* $S^n - D$ *is homologically trivial; i.e.,* $\tilde{H}_i(S^n - D) = 0$ *for all* i.

Theorem 6. *If* S *is an embedded* k*-dimensional sphere in* S^n *for some* $0 \leq k < n$ *then* $S^n - S$ *is homologically equivalent to an* $(n-k-1)$*-dimensional sphere; i.e.,* $\tilde{H}_i(S^n - S)$ *is* $\mathbb{Z}$ *for* $i = n-k-1$ *and* 0 *otherwise.*

Note that these two theorems are homotopically false!

Corollary 7. *(The Jordan Curve Theorem, Veblen 1905) A simple closed curve in the plane separates the plane into exactly two connected components. (In fact, by the same reasoning an embedded* S^{n-1} *in* S^n *separates the latter into exactly two connected components).*

Theorem 8. *(Invariance of Domain) If a subset of* $\mathbb{R}^n$ *is homeomorphic to an open set in* $\mathbb{R}^n$*, then it is an open set in* $\mathbb{R}^n$.

Corollary 9. *If* $M \hookrightarrow N$ *is an embedding of a compact manifold in a connected manifold of the same dimension, then it is a homeomorphism.*

Theorem 10. *(The Borsuk-Ulam Theorem) Every continuous map* $f : S^n \to \mathbb{R}^n$ *identifies a pair of antipodal points.*

Theorem 11. $\mathbb{R}$ *and* $\mathbb{C}$ *are the only commutative division algebras with identity over* $\mathbb{R}$.