THE FULTON-MACPHERSON COMPACTIFICATION

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Let M be a manifold and let A be a finite set.

Definition 1. The open configuration space of A in M is

$$C^o_A(M) := \{\text{injections } \iota : A \to M\}.$$

Definition 2. The compactified configuration space of A in M is

$$C_A(M) := \coprod_{\substack{\{A_1,\dots,A_k\}\\ A=\cup A_\alpha}} \left\{ \left(p_\alpha \in M, c_\alpha \in \tilde{C}_{A_\alpha}(T_{p_\alpha}M)\right)_{\alpha=1}^k : p_\alpha \neq p_\beta \text{ for } \alpha \neq \beta \right\}$$

where if V is a vector space and $|A| \geq 2$,

$$\tilde{C}_A(V) := \coprod_{\substack{\{A_1,\dots,A_k\}\\ A=\cup A_\alpha;\ k\geq 2}} \left\{ \left(v_\alpha \in V, c_\alpha \in \tilde{C}_{A_\alpha}(T_{v_\alpha}V)\right)_{\alpha=1}^k : v_\alpha \neq v_\beta \text{ for } \alpha \neq \beta \right\} \Big/ \begin{matrix}\text{translations and}\\ \text{dilations.}\end{matrix}$$

while if A is a singleton, $\tilde{C}_A(V) := \{\text{a point}\}$.

Theorem 1.
(1) $C_A(M)$ ia a manifold with corners, and if M is compact, so is $C_A(M)$.
(2) If A is a singleton, $C_A(M) = M$. If A is a doubleton, then $C_A(M)$ is isomorphic to $M \times M$ minus a tubular neighborhood of the diagonal $\Delta \subset M \times M$. That is, $C_A(M) = M \times M - V(\Delta)$.
(3) If $B \subset A$ then there is a natural map $C_A(M) \to C_B(M)$. In particular, for every $i, j \in A$ there is a map $\phi_{ij} : C_A(\mathbb{R}^3) \to C_{\{i,j\}}(\mathbb{R}^3) \sim S^2$.
(4) If $f : M \to N$ is a smooth embedding, then there's a natural $f_\star : C_A(M) \to C_A(N)$. □

Now let D be a graph whose set of vertices is A. If two different vertices $a_{0,1} \in A$ are connected by an edge in D, we write $a_0 \overset{D}{-} a_1$. Likewise, if $A_{0,1} \subset A$ are disjoint subsets, we write $A_0 \overset{D}{-} A_1$ if $a_0 \overset{D}{-} a_1$ for some $a_0 \in A_0$ and $a_1 \in A_1$. For any subset A_0 of A we let $D(A_0)$ be the restriction of D to A_0.

Definition 3. The open configuration space of D in M is

$$C^o_D(M) := \{\iota : A \to M :\ \iota(a_0) \neq \iota(a_1) \text{ whenever } a_0 \overset{D}{-} a_1\}.$$

Definition 4. The compactified configuration space of D in M is

$$C_D(M) := \coprod_{\substack{\{A_1,\dots,A_k\}\\ A=\cup A_\alpha\\ \forall\alpha\ D(A_\alpha)\text{ connected}}} \left\{ \left(p_\alpha \in M, c_\alpha \in \tilde{C}_{D(A_\alpha)}(T_{p_\alpha}M)\right)_{\alpha=1}^k : p_\alpha \neq p_\beta \text{ whenever } A_\alpha \overset{D}{-} A_\beta \right\}$$

where if V is a vector space and $|A| \geq 2$,

$$\tilde{C}_D(V) := \coprod_{\substack{\{A_1,\dots,A_k\}\\ A=\cup A_\alpha;\ k\geq 2\\ \forall\alpha\ D(A_\alpha)\text{ connected}}} \left\{ \left(v_\alpha \in V, c_\alpha \in \tilde{C}_{D(A_\alpha)}(T_{v_\alpha}V)\right)_{\alpha=1}^k : v_\alpha \neq v_\beta \text{ whenever } A_\alpha \overset{D}{-} A_\beta \right\} \Big/ \begin{matrix}\text{translations}\\ \text{and}\\ \text{dilations.}\end{matrix}$$

while if A is a singleton, $\tilde{C}_D(V) := \{\text{a point}\}$.

Theorem 2. The obvious parallel of the previous theorem holds. □

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This handout is at available from `http://www.ma.huji.ac.il/~drorbn/classes/0102/FeynmanDiagrams/`.