Chapter 12

1. Find f^{-1} for each of the following f.

(ii) $f(x)=(x-1)^3 \Rightarrow f^{-1}(x)=\sqrt[3]{x}+1$

(iv) $f(x)=\begin{cases}-x^2 & x\ge 0\\ 1-x^3 & x<0\end{cases} \Rightarrow f^{-1}(x)=\begin{cases}\sqrt[3]{1-x} & x>1\\ \sqrt{-x} & x\le 0\end{cases}$

(vi) $f(x)=x+[x] \Rightarrow f^{-1}(x)=x-\dfrac{[x]}{2}$

(viii) $f(x)=\dfrac{x}{1-x^2},\ -1<x<1 \Rightarrow f^{-1}(x)=\begin{cases}0 & x=0\\ \dfrac{\sqrt{1+4x^2}-1}{2x} & x\ne 0\end{cases}$

8. Proof

$g=f^{-1} \Leftrightarrow f(g(x))=x \Rightarrow f'(g(x))\cdot g'(x)=1$ chain rule

$$\therefore g'(x)=\frac{1}{f'(g(x))}$$

$$\because f'(x)=\left(1+x^3\right)^{-\frac{1}{2}}$$

$$\therefore g'(x)=\frac{1}{\left(1+g^3(x)\right)^{-\frac{1}{2}}}=\left(1+g^3(x)\right)^{\frac{1}{2}}$$

$$\therefore g''(x)=\frac{1}{2}\left(1+g^3(x)\right)^{-\frac{1}{2}}\cdot 3g^2(x)\cdot g'(x)=\frac{3}{2}g^2(x)\cdot\left(1+g^3(x)\right)^{-\frac{1}{2}}\cdot\left(1+g^3(x)\right)^{\frac{1}{2}}=\frac{3}{2}g^2(x)$$

Q.E.D

15. Solution

Look y as a function of x, derivative the equation: $3x^3+4x^2y-xy^2+2y^3=4$, then:

$$\frac{d(3x^3)}{dx}+\frac{d(4x^2y)}{dx}+\frac{d(-xy^2)}{dx}+\frac{d(2y^3)}{dx}=\frac{4}{dx}\Leftrightarrow$$

$$(9x^2)+\left(4x^2\frac{dy}{dx}+8x\right)+\left(-y^2-x\cdot 2y\cdot\frac{dy}{dx}\right)+\left(6y^2\frac{dy}{dx}\right)=0$$

since the point $(-1,1)$ is on the curve, thus:

$$(9x^2)+\left(4x^2\frac{dy}{dx}+8x\right)+\left(-y^2-x\cdot 2y\cdot\frac{dy}{dx}\right)+\left(6y^2\frac{dy}{dx}\right)=0\Leftrightarrow$$

$$9+4\frac{dy}{dx}-8-1+2\frac{dy}{dx}+6\frac{dy}{dx}=0\Leftrightarrow 12\frac{dy}{dx}=0\Leftrightarrow\frac{dy}{dx}=0$$

$\therefore$ the equation of the tangent line at point $(-1,1)$ is $y=1$.

20. Solution

let $g = f^{-1} \Leftrightarrow f(g(x)) = x \Rightarrow f'(g(x)) \cdot g'(x) = 1$ chain rule

$$\therefore g'(x) = \frac{1}{f'(g(x))} \Rightarrow g''(x) = -\frac{(f'(g(x)))'}{(f'(g(x)))^2} = -\frac{f''(g(x))(g'(x))}{(f'(g(x)))^2}$$

$$= -\frac{f''(g(x))}{(f'(g(x)))^3}$$

$$\therefore (f^{-1})''(x) = -\frac{f''(f^{-1}(x))}{(f'(f^{-1}(x)))^3}$$