Math 157 Analysis I — Solution of Term Exam 1

web version:
http://www.math.toronto.edu/~drorbn/classes/0304/157AnalysisI/TermExam1/Solution.html

Problem 1. All that is known about the angle α is that $\tan\frac{\alpha}{2} = \sqrt{2}$. Can you find $\sin\alpha$ and $\cos\alpha$? Explain your reasoning in full detail.
Solution. (Graded by C.-N. (J.) Hung) In class we wrote the formula $\sin 2\beta = 2\sin\beta\cos\beta$. Also using $\sin^2\beta + \cos^2\beta = 1$ and taking $\beta = \frac{\alpha}{2}$ we get

$$\sin\alpha = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}}.$$

Dividing the numerator and denominator by $\cos^2\frac{\alpha}{2}$ this becomes

$$\frac{2\tan\frac{\alpha}{2}}{\tan^2\frac{\alpha}{2} + 1} = \frac{2\sqrt{2}}{\sqrt{2}^2 + 1} = \frac{2\sqrt{2}}{3}.$$

Likewise using $\cos 2\beta = \cos^2\beta - \sin^2\beta$ we get

$$\cos\alpha = \frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}} = \frac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} = \frac{1 - \sqrt{2}^2}{1 + \sqrt{2}^2} = -\frac{1}{3}.$$

Problem 2.

1. State the definition of the natural numbers.
2. Prove that every natural number n has the property that whenever m is natural, so is $m + n$.

Solution. (Graded by V. Tipu)

1. The set of natural numbers $\mathbb{N}$ is the smallest set of numbers for which
   - $1 \in \mathbb{N}$,
   - if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$.

   Alternatively, the set of natural numbers $\mathbb{N}$ is the intersection of all sets I of numbers satisfying
   - $1 \in I$,
   - if $n \in I$ then $n + 1 \in I$.
2. Let $P(n)$ be the assertion "whenever m is natural, so is $m + n$". We prove $P(n)$ by induction on n:

(a) $P(1)$ asserts that "whenever m is natural, so is $m+1$". This is true by the second bullet in the definition of $\mathbb{N}$.

(b) Assume $P(n)$, that is, assume that whenever m is natural, so is $m+n$. Let m be a natural number. Then $m+(n+1) = (m+n)+1$ is a natural number because by $P(n)$ the number $m+n$ is natural and because adding one to a natural number gives a natural number by the second bullet in the definition of $\mathbb{N}$. So we have shown that whenever m is natural so is $m+(n+1)$, and this is the assertion $P(n+1)$.

Problem 3. Recall that a function g is called "even" if $g(x) = g(-x)$ for all x and "odd" if $g(-x) = -g(x)$ for all x, and let f be some arbitrary function.

1. Find an even function E and an odd function O so that $f = E + O$.

2. Show that if $f = E_1 + O_1 = E_2 + O_2$ where E_1 and E_2 are even and O_1 and O_2 are odd, then $E_1 = E_2$ and $O_1 = O_2$.

Solution. (Graded by C. Ivanescu)

1. Set $E(x) = \frac{1}{2}(f(x) + f(-x))$ and $O(x) = \frac{1}{2}(f(x) - f(-x))$. Then $E(x) + O(x) = \frac{1}{2}(f(x) + f(-x) + f(x) - f(-x)) = \frac{1}{2}(2f(x)) = f(x)$ while $E(-x) = \frac{1}{2}(f(-x) + f(-(-x))) = \frac{1}{2}(f(x) + f(-x)) = E(x)$ (so E is even) and $O(-x) = \frac{1}{2}(f(-x) - f(-(-x))) = -\frac{1}{2}(f(x) - f(-x)) = -O(x)$ (so O is odd).

2. Assume $f = E + O$ where E is even and O is odd. Then

$$f(x)+f(-x) = E(x)+O(x)+E(-x)+O(-x) = E(x)+O(x)+E(x)-O(x) = 2E(x).$$

So *necessarily* $E(x) = \frac{1}{2}(f(x) + f(-x))$. Now if $f = E_1 + O_1 = E_2 + O_2$ as above, then both E_1 and E_2 can play the role of E in this argument, so they are both equal to $\frac{1}{2}(f(x) + f(-x))$ and in particular they equal each other. Likewise,

$$f(x) - f(-x) = E(x)+O(x) - E(-x) - O(-x) = E(x)+O(x) - E(x)+O(x) = 2O(x)$$

and arguing like before, $O_1(x) = \frac{1}{2}(f(x) - f(-x)) = O_2(x)$.

Problem 4. Sketch, to the best of your understanding, the graph of the function

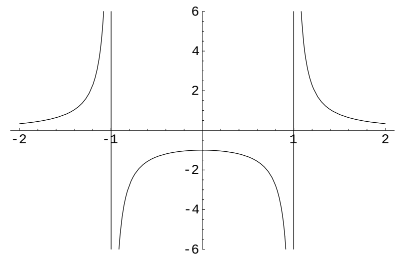
$$f(x) = \frac{1}{x^2 - 1}.$$

(What happens for x near 0? Near ± 1? For large x? Is the graph symmetric? Does it appear to have a peak somewhere?)

Solution. (Graded by C. Ivanescu)

If $|x| > 1$ then $x^2 - 1 > 0$ and so $\frac{1}{x^2-1} > 0$; furthermore, the larger $|x|$ is (while $|x| > 1$), the larger $x^2 - 1$ is and hence the smaller $\frac{1}{x^2-1}$ is. When $|x|$ approaches 1 from above, $x^2 - 1$ approaches 0 from above and hence $\frac{1}{x^2-1}$ becomes larger and larger. If $|x| < 1$ the $x^2 - 1 < 0$ and so $\frac{1}{x^2-1} < 0$. When $x = 0$, $f(x) = -1$ and when $|x|$ approaches 1 from below, x^2

approaches 1 from below and $x^2 - 1$ approaches 0 from below, and so $\frac{1}{x^2-1}$ becomes more and more negative. In summary, the graph looks something like:



Problem 5.

1. Suppose that $f(x) \leq g(x)$ for all x, and that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Prove that $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

2. Suppose that $f(x) < g(x)$ for all x, and that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Is it always true that $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$? (If you think it's always true, write a proof. If you think it isn't always true, provide a counterexample).

Solution. (Graded by C.-N. (J.) Hung)

1. Let $l = \lim_{x\to a} f(x)$ and $m = \lim_{x\to a} g(x)$ and assume by contradiction that $l > m$; that is, that $\epsilon := \frac{l-m}{2} > 0$. Use the existence of the two limits to find $\delta_1 > 0$ and $\delta_2 > 0$ so that
$$0 < |x-a| < \delta_1 \Longrightarrow |f(x) - l| < \epsilon$$
and
$$0 < |x-a| < \delta_2 \Longrightarrow |g(x) - m| < \epsilon.$$
Now choose some specific $x \neq a$ for which both $|x-a| < \delta_1$ and $|x-a| < \delta_2$. But then $|f(x) - l| < \epsilon$ and so $f(x) > l - \epsilon$ while $|g(x) - m| < \epsilon$ and so $g(x) < m + \epsilon$. Therefore remembering that $f(x) \leq g(x)$ for all x we get
$$l - \epsilon < f(x) \leq g(x) < m + \epsilon$$
or
$$l - \frac{l-m}{2} < m + \frac{l-m}{2}$$
or
$$\frac{m+l}{2} < \frac{m+l}{2}$$
which is a contradiction. Thus the assumption that $l > m$ must be incorrect and thus $m \leq l$.

2. Take $f(x) = 0$ for all x and $g(x) = x^2$ for all $x \neq 0$ and $g(0) = 157$. Then $f(x) < g(x)$ for all x but $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$. So it isn't always true that if $f(x) < g(x)$ for all x and the limits exist, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

The results. 105 students took the exam; the average grade was 67.19, the median was 70 and the standard deviation was 21.12.