Dror Bar-Natan: Classes: 2003-04: Math 157 - Analysis I:

Visualization

web version:
http://www.math.toronto.edu/~drorbn/classes/0304/157AnalysisI/Visualization/Visualization.html

Our task for this week is to master the axiomatically meaningless task of visualization of numbers and functions. We will learn how to interpret graphically all of the following:

1. A number a, the order relation $a < b$ and the absolute value of a difference $|a - b|$.
2. Intervals such as $(a, b) := \{x : a < x < b\}$, $[a, b) := \{x : a \leq x < b\}$, $[a, b] := \{x : a \leq x \leq b\}$, $(a, \infty) := \{x : x > a\}$ and $(-\infty, a] := \{x : x \leq a\}$.
3. A point (a, b) in the plane. (Notice the sad clash of notation).
4. The graphs of the functions $f_1(x) = c$, $f_2(x) = cx$ and $f_3(x) = cx + d$.
5. The Euclidean distance function $d((a, b), (c, d)) := \sqrt{(a-c)^2 + (b-d)^2}$.
6. The parabola $y = x^2$ and the graphs of $f(x) = x^n$ for several n's.
7. The graphs of $f_1(x) = \frac{1}{x}$, $f_2(x) = \frac{1}{x^2}$, $f_3(x) = \frac{1}{1+x^2}$ and $f_4(x) = \frac{x}{1+x^2}$.
8. The graphs of $f_1(x) = \sin x$, $f_2(x) = \sin\frac{1}{x}$, $f_3(x) = x\sin\frac{1}{x}$ and $f_4(x) = x^2\sin\frac{1}{x}$.
9. The graphs of $f_1(x) = \begin{cases} x^2 & x < 1 \\ 2 & x \geq 1 \end{cases}$, $f_2(x) = \begin{cases} x^2 & x \leq 1 \\ 2 & x > 1 \end{cases}$ and $f_3(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$.
10. The circle $(x-a)^2 + (y-b)^2 = r^2$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Just for fun. For $x \in [0, 1]$, the number $f(x)$ is defined to be the result of the following process: Write x in binary, replace every 1 in the resulting expansion by a 2, and interpret the result as a number written in base 3. For example, $x = \frac{1}{3} = 0.01010101_2\ldots \to 0.02020202_3\ldots = \frac{1}{4} = f(x)$.

- Draw the graph of f.
- Draw the range of f as a subset of $[0, 1]$. (The answer, called "the Cantor set" plays a major role in much of analysis and in particular in the theory of fractals. In some sense its dimension is the irrational number $\frac{\log 2}{\log 3}$.)