Term Exam 4

University of Toronto, March 21, 2005

Solve all of the following 4 problems. Each problem is worth 25 points. You have an hour and 50 minutes. Write your answers in the Term Exam notebooks provided and **not** on this page.

Allowed Material: Any calculating device that is not capable of displaying text.

Web version: http://www.math.toronto.edu/~drorbn/classes/0405/157AnalysisI/TE4/TE4.html

Good Luck!

Problem 1. Agents of CSIS have secretly developed a function $E(x)$ that has the following properties:

- $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$.
- $E(0) = 1$
- E is differentiable at 0 and $E'(0) = 1$.

Prove the following:

1. E is everywhere differentiable and $E' = E$.
2. $E(x) = e^x$ for all $x \in \mathbb{R}$. The only lemma you may assume is that if a function f satisfies $f'(x) = 0$ for all x then f is a constant function.

Problem 2. Compute the following integrals: (a few lines of justification are expected in each case, not just the end result.)

1. $\displaystyle\int \frac{x^2+1}{x+2} dx.$
2. $\displaystyle\int e^{ax} \sin bx \, dx$ (assume that $a, b \in \mathbb{R}$ and that $a \neq 0$ and $b \neq 0$).
3. $\displaystyle\int x \log \sqrt{1+x^2} \, dx.$
4. $\displaystyle\int_0^\infty e^{-x} \, dx.$ (This, of course, is $\displaystyle\lim_{T\to\infty} \int_0^T e^{-x} \, dx$).

Problem 3.

1. State (without proof) the formula for the surface area of an object defined by spinning the graph of a function $y = f(x)$ (for $a \le x \le b$) around the x axis.
2. Compute the surface area of a sphere of radius 1.

Problem 4.

1. State and prove the remainder formula for Taylor polynomials (it is sufficient to discuss just one form for the remainder, no need to mention all the available forms).
2. It is well known (and need not be reproven here) that the nth Taylor polynomial $P_n = P_{n,0,e^x}$ of e^x around 0 is given by $P_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$. It is also well known (and need not be reproven here) that factorials grow faster than exponentials, so for any fixed c we have $\lim_{n\to\infty} c^n/n! = 0$. Show that for large enough n,

$$\left|e^{157} - P_n(157)\right| < \frac{1}{157}.$$