M.Sc. Math Workshop — Assignment #2

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(7) Prove: If $\lambda > 0$ is irrational then there exists a continuous function $\phi : [0,1] \to [0,1]$ so that for every $\epsilon > 0$ and every every continuous function $f : [0,1]\times[0,1] \to \mathbf{R}$ there exists a continuous function $g : [0, 1+\lambda] \to \mathbf{R}$ so that $|f(x,y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on a set of area at least $1 - \epsilon$ in $[0,1] \times [0,1]$. (Notice the different order of the quantifiers relative to Q2).

(8) Can you completely cover a disk of diameter 100 with 99 rectangles of sides 100×1?

(9) Let A and B be two $n \times n$ matrices over $\mathbf{C}$.

(a) Prove that $[A, B] \stackrel{\text{def}}{=} AB - BA \neq I$.

(b) Prove that if $[A, [A, B]] = 0$, then $[A, B]$ is nilpotent.

(10) Find a 2-variable polynomial that is always positive on $\mathbf{R}^2$, and has exactly two critical points, both of which are minima.

(11) Let $C_n = \{(z_1, \dots, z_n) \in \mathbf{C}^n : \forall 1 \leq < j \leq n,\ z_i \neq z_j$ be the configuration space of n distinct points in the complex plane $\mathbf{C}$, and let the nth pure braid group $PB_n = \pi_1(C_n)$ be the fundamental group of C_n. Prove that $PB_n \simeq PB_{n-1} \ltimes F_{n-1}$ where F_{n-1} denotes the free group on $n-1$ generators. Deduce that $PB_n \simeq F_1 \ltimes F_2 \ltimes \cdots \ltimes F_{n-1}$.

(12) Let $A_n = \{1, 2, 3\}^n$ and let $A = A_1$. A function $f : A_n \to A$ is called *injective*, if whenever x and y are different, $f(x) \neq f(y)$. Obviously, injective functions exist only if $n = 1$, and in this case, they are simply permutations $\pi : A \to A$. A function $f : A_n \to A$ is called *weakly injective*, if whenever x and y are *totally different*, meaning that $x_i \neq y_i$ for *all* $1 \leq i \leq n$, one has $f(x) \neq f(y)$. Prove that if $f : A_n \to A$ is weakly injective then for some permutation $\pi : A \to A$ and $1 \leq i \leq n$, one has $f(x) = \pi(x_i)$ for all $x \in A_n$.

(13) f is a real valued function on the reals, and it is known that at any point at least one derivative of f vanishes (possibly different derivatives at different points). Prove that f is a polynomial.

(14) What is the configuration space of the machine M_3 in the picture below?

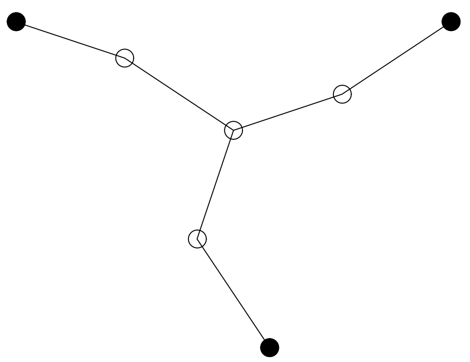

Figure 3. The machine M_3.