M.Sc. Math Workshop — Assignment #6

HUJI Spring 1998

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(21) Give two proofs that the configuration space of a 6-legged roach (such as the one in Figure 4) is an oriented surface of genus 17:
- Using Euler characteristics,
- And by a direct cut-and-paste argument.

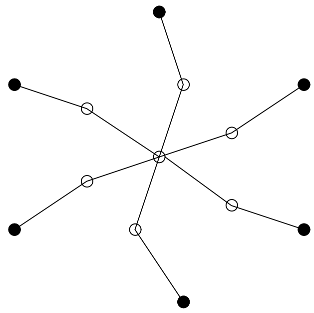
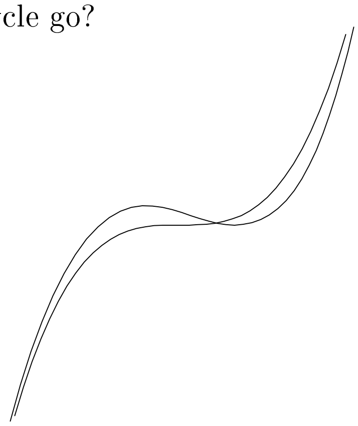


Figure 4. A roach. Don't worry! It is safely attached to this page and it cannot jump out.

(22) Which way did the bicycle go?



(23) A *smooth pre-knot* is a smooth (C^∞) embedding of S^1 (the circle) in $\mathbf{R}^3$. We say that two smooth pre-knots are smoothly equivalent if there is a smooth homotopy between them, which is also an embedding at all intermediate times. A *smooth knot* is an equivalence class of smooth pre-knots, modulo smooth equivalence. Similarly we can define *continuous knots*. Are the two notions equivalent?

(24) We say that two smooth pre-knots $\gamma_{1,2} : S^1 \to \mathbf{R}^3$ are smoothly ambient-equivalent if there exists a diffeomorphism (smooth bijection with a smooth inverse) $f : \mathbf{R}^3 \to \mathbf{R}^3$ so that $\gamma_2 = f \circ \gamma_1$. A *smooth ambient knot* is an equivalence class of smooth pre-knots modulo smooth ambient-equivalence. Similarly we can define *continuous ambient knots*. Are the two notions equivalent?

(25) Are smooth ambient knots equivalent to smooth knots?

(26) A *polygonal pre-knot* is simply a polygon embedded in $\mathbf{R}^3$. Two polygonal pre-knots are called Δ-equivalent if they differ by a sequence of triangle moves such as in Figure 5, in which no other parts of the polygon pass through the triangle. A

polygonal knot is an equivalence class of polygonal pre-knots modulo Δ-equivalence. Are polygonal knots equivalent to smooth knots?

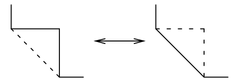


Figure 5. A triangle move.

(27) Come up with a reasonable notion of "a knot projection" (in $\mathbf{R}^2$, of a knot in $\mathbf{R}^3$). We say that two knot projections are R-equivalent if they differ by a sequence of "Reidemeister" moves of kinds $R1$, $R2$, and $R3$, as shown in Figure 6. Prove that the set of polygonal knots is equivalent to the set of knot projections modulo R-equivalence.

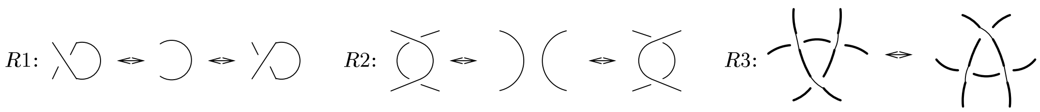


Figure 6. The three Reidemeister moves.

(28) An "interval" in a knot projection is precisely what you think it is. For example, the standard projection of the trefoil knot is made of three intervals. If P is knot projection, define $\nu(P)$ to be YES if the intervals of P can be colored with three colors so that

- all three colors are used,
- and in each crossings, the three intervals involved are either all colored the same way or in three different colors.

Otherwise set $\nu(P)$ to be NO. Prove that ν is a knot invariant (namely, it is invariant under the three Reidemeister moves), and compute it on the unknot and on the trefoil knot.