M.Sc. Math Workshop — Assignment #7

HUJI Spring 1998

Dror Bar-Natan

(29) Prove that non-convex quadrangles cannot tile a convex region.

(30) Zoologists have studied the hopping habits of frogs and determined that frogs can only hop over other frogs, and when they do so they land equally far from the frog they've just hopped over as they've been before, but on its opposite side. Prove that four frogs, initially on the corners of a regular square, cannot hop over each other a finite number of times and at the end form a larger regular square.

(31) Can you place uncountably many disjoint Y's in the plane? A Y is a union of three short straight lines that meet in one point. The legs of your Y's can be of different sizes and can point to different directions.

(32) Can you cover a set of non-zero volume in $\mathbf{R}^3$ with disjoint geometric circles of unit radius? (A geometric circle of unit radius is a rotation of a translation of the standard unit circle in the plane).

(33) Can you cover $\mathbf{R}^3$ with disjoint geometric circles (not necessarily of the same radius)?

(34) Prove: If $\lambda > 0$ is irrational and $\epsilon > 0$ then there exists 5 continuous functions $\phi_i : [0,1] \to [0,1]$ $(1 \le i \le 5)$ so that for every continuous function $f : [0,1] \times [0,1] \to \mathbf{R}$ there exists a continuous function $g : [0, 1+\lambda] \to \mathbf{R}$ so that

$$f(x,y) = \sum_{i=1}^{5} g(\phi_i(x) + \lambda\phi_i(y))$$

for every $x, y \in [0,1]$.

(35) Find three diagonalizable but non-diagonal 4×4 matrices A, B, and C, so that A commutes with B, B commutes with C, but C doesn't commute with A. Prove that the same cannot be done with 2×2 matrices. How about 3×3?

(36) Let $(\mathcal{H}, v)$ be a finite dimensional quantum probability space.

(a) Let A and B be commuting random variables on $\mathcal{H}$. Prove that there exists a unique probability measure μ_{AB} on $\mathbf{R}^2$ so that $\int_{\mathbf{R}^2} x^n y^m \, d\mu_{AB}(x,y) = \langle v, A^n B^m v\rangle$.

(b) In the light of Q16, explain why it makes sense to call μ_{AB} "the joint distribution of A and B".

(c) Find some finite dimensional quantum probability space $(\mathcal{H}, v)$ along with four random variables A, B, C, D on it, so that the joint distributions μ_{AB}, μ_{BC}, μ_{CD}, and μ_{DA} exist, and so that $P(A = B) = \frac{3}{4}$, $P(B = C) = \frac{3}{4}$, $P(C = D) = \frac{3}{4}$, but $P(D = A) = 0$.