FINITE TYPE INVARIANTS

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Abstract. This is an overview article on finite type invariants, written for the *Encyclopedia of Mathematical Physics*.

1. Introduction

Knots belong to sailors and climbers and upon further reflection, perhaps also to geometers, topologists or combinatorialists. Surprisingly, throughout the 1980s it became apparent that knots are also closely related to several other branches of mathematics in general and mathematical physics in particular. Many of these connections (though not all!) factor through the notion of "finite type invariants" (aka "Vassiliev" or "Goussarov-Vassiliev" invariants) [Go1, Go2, Va1, Va2, BL, Ko1, BN2].

Let V be an arbitrary invariant of oriented knots in oriented space with values in some Abelian group A. Extend V to be an invariant of 1-singular knots, knots that may have a single singularity that locally looks like a double point , using the formula

$$V(\text{}) = V(\text{}) - V(\text{}). \tag{1}$$

Further extend V to the set $\mathcal{K}^m$ of m-singular knots (knots with m double points) by repeatedly using (1).

Definition 1. We say that V is of type m if its extension $V|_{\mathcal{K}^{m+1}}$ to $(m+1)$-singular knots vanishes identically. We say that V is of finite type if it is of type m for some m.

Repeated differences are similar to repeated derivatives and hence it is fair to think of the definition of $V|_{\mathcal{K}^m}$ as repeated differentiation. With this in mind, the above definition imitates the definition of polynomials of degree m. Hence finite type invariants can be thought of as "polynomials" on the space of knots.

As we shall see below, finite type invariants are plenty and powerful and they carry a rich algebraic structure and are deeply related to Lie algebras (Section 2). There are several constructions for a "universal finite type invariant" and those are related to conformal field theory, the Chern-Simons-Witten topological quantum field theory and Drinfel'd's theory of associators and quasi-Hopf algebras (Section 3). Finite type invariants have been studied extensively (Section 4) and generalized in several directions (Section 5). But the first question on finite type invariants remains unanswered:

Problem 2. *Honest polynomials are dense in the space of functions. Are finite type invariants dense within the space of all knot invariants? Do they separate knots?*

In a similar way one may define finite type invariants of framed knots (and ask the same questions).

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2. Basic facts

2.1. **Classical knot polynomials.** The first (non trivial!) thing to notice is that there are plenty of finite type invariants and they are at least as powerful as all the standard knot polynomials[1] combined:

Theorem 3. [BN2, BL] *Let $J(K)(q)$ be the Jones polynomial of a knot K (it is a Laurent polynomial in a variable q). Consider the power series expansion $J(K)(e^x) = \sum_{m=0}^{\infty} V_m(K)x^m$. Then each coefficient $V_m(K)$ is a finite type knot invariant. (And thus the Jones polynomial can be reconstructed from finite type information).*

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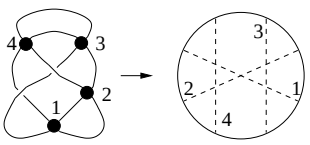
[1]Finite type invariants are like polynomials on the space of knots; the standard phrase "knot polynomials" refers to a different thing — knot invariants with polynomial values.

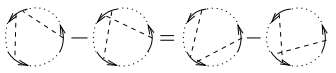
A similar theorem holds for the Alexander-Conway, HOMFLY-PT and Kauffman polynomials [BN2], and indeed, for arbitrary Reshetikhin-Turaev invariants [RT], [Lin1]. Though it is still unknown if the signature of a knot can be expressed in terms of its finite type invariants.

2.2. **Chord diagrams and the Fundamental Theorem.** The top derivatives of a multi-variable polynomial form a system of constants that determine that polynomial up to polynomials of lower degree. Likewise the mth derivative $V^{(m)} := V\left(\times^{m}\right)$ of a type m invariant V is a constant (for $V\left(\times^{m}\right) - V\left(\times^{m}\right) = V\left(\times^{m+1}\right) = 0$ so $V^{(m)}$ is blind to 3D topology) and likewise $V^{(m)}$ determines V up to invariants of lower type. Hence a primary tool in the study of finite type invariants is the study of the "top derivative" $V^{(m)}$, also known as "the weight system of V".



Blind to 3D topology, $V^{(m)}$ only sees the combinatorics of the circle that parametrizes an m-singular knot. On this circle there are m pairs of points that are pairwise identified in the image; standardly one indicates those by drawing a circle with m chords marked (an "m-chord diagram") as on the right.

Definition 4. Let $\mathcal{D}_m$ denote the space of all formal linear combinations with rational coefficients of m-chord diagrams. Let $\mathcal{A}^r_m$ be the quotient of $\mathcal{D}_m$ by all $4T$ and FI relations as drawn below (full details in e.g. [BN2]), and let $\hat{\mathcal{A}}^r$ be the graded completion of $\mathcal{A} := \bigoplus_m \mathcal{A}^r_m$. Let $\mathcal{A}_m$, $\mathcal{A}$ and $\hat{\mathcal{A}}$ be the same as $\mathcal{A}^r_m$, $\mathcal{A}^r$ and $\hat{\mathcal{A}}^r$ but without imposing the FI relations.

$4T$:  FI :

Theorem 5. *(The Fundamental Theorem)*
- *(Easy part,* [Va1, Go1, BL]*). If V is a rational valued type m invariant then $V^{(m)}$ defines a linear functional on $\mathcal{A}^r_m$. If in addition $V^{(m)} \equiv 0$, then V is of type $m-1$.*
- *(Hard part,* [Ko1] *and Section 3). For any linear functional W on $\mathcal{A}^r_m$ there is a rational valued type m invariant V so that $V^{(m)} = W$.*

Thus to a large extent the study of finite type invariants is reduced to the finite (though super exponential in m) algebraic study of $\mathcal{A}^r_m$. A similar theorem reduces the study of finite type invariants of framed knots to the study of $\mathcal{A}_m$.

2.3. **The structure of $\mathcal{A}$.** Knots can be multiplied (the "connected sum" operation) and knot invariants can be multiplied. This structure interacts well with finite type invariants and induces the following structure on $\mathcal{A}^r$ and $\mathcal{A}$:

Theorem 6. [Ko1, BN2, Wil, CDL] *$\mathcal{A}^r$ and $\mathcal{A}$ are commutative and cocommutative graded bialgebras (i.e., each carries a commutative product and a compatible cocommutative coproduct). Thus both $\mathcal{A}^r$ and $\mathcal{A}$ are graded polynomial algebras over their spaces of primitives, $\mathcal{P}^r = \oplus_m \mathcal{P}^r_m$ and $\mathcal{P} = \oplus_m \mathcal{P}_m$.*

Framed knots differ from knots only by a single integer parameter (the "self linking", itself a type 1 invariant). Thus $\mathcal{P}^r$ and $\mathcal{P}$ are also closely related.

Theorem 7. [BN2] *$\mathcal{P} = \mathcal{P}^r \oplus \langle\theta\rangle$, where θ is the unique 1-chord diagram* .

2.4. **Bounds and computational results.** The following table (taken from [BN2, Kn]) shows the number of type m invariants of knots and framed knots modulo type $m-1$ invariants ($\dim \mathcal{A}^r_m$ and $\dim \mathcal{A}_m$) and the number of multiplicative generators of the algebra $\mathcal{A}$ in degree m ($\dim \mathcal{P}_m$) for $m \le 12$. Some further tabulated results are in [BN5].

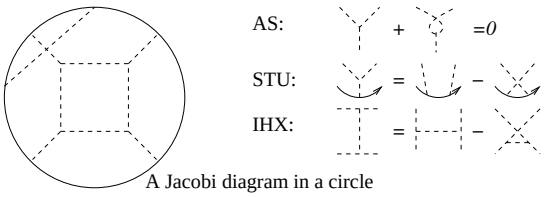
m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

Little is known about these dimensions for large m. There is an explicit conjecture in [Br] but no progress has been made in the direction of proving or disproving it. The best asymptotic bounds available are:

Theorem 8. *For large m, $\dim \mathcal{P}_m > e^{c\sqrt{m}}$ (for any fixed $c < \pi\sqrt{\frac{2}{3}}$)* [Da, Ko3] *and $\dim \mathcal{A}_m < 6^m m!\sqrt{m}/\pi^{2m}$* [Sto, Za].

2.5. **Jacobi diagrams and the relation with Lie algebras.** Much of the richness of finite type invariants stems from their relationship with Lie algebras. Theorem 9 below suggests this relationship on an abstract level, Theorem 10 makes that relationship concrete and Theorem 12 makes it a bit deeper.

Theorem 9. [BN2] *The algebra $\mathcal{A}$ is isomorphic to the algebra $\mathcal{A}^t$ generated by "Jacobi diagrams in a circle" (chord diagrams that are also allowed to have oriented internal trivalent vertices) modulo the AS, STU and IHX relations. See the figure below.*



A Jacobi diagram in a circle

Thinking of trivalent vertices as graphical analogs of the Lie bracket, the AS relation become the anti-commutativity of the bracket, STU become the equation $[x,y] = xy - yx$ and IHX becomes the Jacobi identity. This analogy is made concrete within the proof of the following:

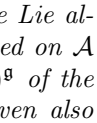
Theorem 10. [BN2] *Given a finite dimensional metrized Lie algebra $\mathfrak{g}$ (e.g., any semi-simple Lie algebra) there is a map $\mathcal{T}_\mathfrak{g} : \mathcal{A} \to \mathcal{U}(\mathfrak{g})^\mathfrak{g}$ defined on $\mathcal{A}$ and taking values in the invariant part $\mathcal{U}(\mathfrak{g})^\mathfrak{g}$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. Given also a finite dimensional representation R of $\mathfrak{g}$ there is a linear functional $W_{\mathfrak{g},R} : \mathcal{A} \to \mathbb{Q}$.*

The last assertion along with Theorem 5 show that associated with any $\mathfrak{g}$, R and m there is a weight system and hence a knot invariant. Thus knots are unexpectedly linked with Lie algebras.

The hope [BN2] that all finite type invariants arise in this way was dashed by [Vo1, Vo2, Lie1]. But finite type invariants that do not arise in this way remain rare and not well understood.

The Poincaré-Birkhoff-Witt (PBW) theorem of the theory of Lie algebras says that the obvious "symmetrization" map $\chi_\mathfrak{g} : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ from the symmetric algebra $\mathcal{S}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ to its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a $\mathfrak{g}$-module isomorphism. The following definition and theorem form a diagrammatic counterpart of this theorem:

Definition 11. Let $\mathcal{B}$ be the space of formal linear combinations of "free Jacobi diagrams" (Jacobi diagrams as before, but with unmarked univalent ends ("legs") replacing the circle; see an example on the right), modulo the AS and IHX relations of before. Let $\chi : \mathcal{B} \to \mathcal{A}$ be the symmetrization map which maps a k-legged free Jacobi diagram to the average of the $k!$ ways of planting these legs along a circle.



Theorem 12. *(diagrammatic PBW,* [Ko1, BN2]*) χ is an isomorphism of vector spaces. Furthermore, fixing a metrized $\mathfrak{g}$ there is a commutative square as on the right.*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\chi} & \mathcal{A} \\ \downarrow{\scriptstyle \mathcal{T}_\mathfrak{g}} & & \downarrow{\scriptstyle \mathcal{T}_\mathfrak{g}} \\ \mathcal{S}(\mathfrak{g}) & \xrightarrow{\chi_\mathfrak{g}} & \mathcal{U}(\mathfrak{g}) \end{array}$$

Note that $\mathcal{B}$ can be graded (by half the number of vertices in a Jacobi diagram) and that χ respects degrees so it extends to an isomorphism $\chi : \hat{\mathcal{B}} \to \hat{\mathcal{A}}$ of graded completions.

3. Proofs of the Fundamental Theorem

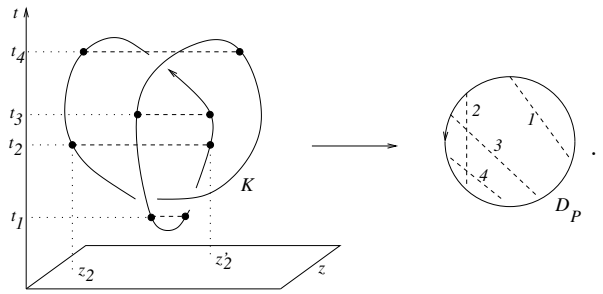
The heart of all known proofs of Theorem 5 is always a construction of a "universal finite type invariant" (see below); it is simple to show that the existence of a universal finite type invariant is equivalent to Theorem 5.

Definition 13. A universal finite type invariant is a map $Z : \{\text{knots}\} \to \hat{\mathcal{A}}^r$ whose extension to singular knots satisfies $Z(K) = D +$ (higher degrees) whenever a singular knot K and a chord diagram D are related as in Section 2.2.

3.1. **The Kontsevich Integral.** The first construction of a universal finite type invariant was given by Kontsevich [Ko1] (see also [BN2, CD]). It is known as "the Kontsevich Integral" and up to a normalization factor it is given by[2]

$$Z_1(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \sum\!\!\!\!\!\!\int_{\substack{t_1<\ldots<t_m \\ P=\{(z_i,z_i')\}}} (-1)^{\#P_\downarrow} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'},$$

where the relationship between the knot K, the pairing P, the real variables t_i, the complex variables z_i and z_i' and the chord diagram D_P is summarized by the figure



The Kontsevich Integral arises from studying the holonomy of the Knizhnik-Zamolodchikov equation of conformal field theory [KZ]. When evaluating Z_1 one encounters multiple ζ-numbers [LM1] in a substantial way, and the proof that the end result is rational is quite involved [LM2] and relies on deep results about associators and quasitriangular Quasi-Hopf algebras [Dr1, Dr2]. Employing the same techniques, in [LM2] it is also shown that the composition of $W_{\mathfrak{g},R} \circ Z_1$ precisely reproduces the Reshetikhin Turaev invariants [RT].

[2]The symbol $\sum\!\!\!\!\!\!\int$ means "sum over all discrete variables and integrate over all continuous variables".

3.2. **Perturbative Chern-Simons-Witten theory and configuration space integrals.** Historically the first approach to the construction of a universal finite type invariant was to use perturbation theory with the Chern-Simons-Witten topological quantum field theory; this is also how the relationship with Lie algebras first arose [BN1]. But taming the integrals involved turned out to be difficult and working constructions using this approach appeared only a bit later [BoTa, Th1, AF].

In short, one writes a perturbative expansion for the large k asymptotics of the Chern-Simons-Witten path integral for some metrized Lie algebra $\mathfrak{g}$ with a Wilson loop in some representation R of $\mathfrak{g}$,

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, tr_R hol_K(A) \exp\left[\tfrac{ik}{4\pi}\int_{\mathbb{R}^3} \mathrm{tr}\big(A\wedge dA+\tfrac{2}{3}A\wedge A\wedge A\big)\right].$$

The result is of the form

$$\sum_{\substack{D\text{: Feynman}\\ \text{diagram}}} W_{\mathfrak{g}(D),R} \oint\!\!\!\!\oint \mathcal{E}(D)$$

where $\mathcal{E}(D)$ is a very messy integral expression and the diagrams D as well as the weights $W_{\mathfrak{g}(D),R}$ are as in Section 2.5. Replacing $W_{\mathfrak{g}(D),R}$ by simply D in the above formula we get an expression with values in $\hat{\mathcal{A}}$:

$$Z_2(K) := \sum_D D \oint\!\!\!\!\oint \mathcal{E}(D) \in \hat{\mathcal{A}}.$$

For formal reasons $Z_2(K)$ ought to be a universal finite type invariant, and after much work taming the $\mathcal{E}(D)$ factors and after multiplying by a further framing-dependent renormalization term Z^{anomaly}, the result is indeed a universal finite type invariant.

Upon further inspection the $\mathcal{E}(D)$ factors can be reinterpreted as integrals of certain spherical volume forms on certain (compactified) configuration spaces [BoTa]. These integrals can be further interpreted as counting certain "tinker toy constructions" built on top of K [Th1]. The latter viewpoint makes the construction of Z_2 visually appealing [BN9], but there is no satisfactory writeup of this perspective yet.

We note that the precise form of the renormalization term Z^{anomaly} remains an open problem. An appealing conjecture is that $Z^{\mathrm{anomaly}} = \exp\frac{1}{2}\circlearrowleft$. If this is true then $Z_2 = Z_1$ [Po]; but the conjecture is only verified up to degree 6 [Les] (there's also an unconfirmed verification to all orders [Ya]).

The most important open problem about perturbative Chern-Simons-Witten theory is not directly about finite type invariants, but it is nevertheless worthwhile to recall it here:

Problem 14. *Does the perturbative expansion of the Chern-Simons-Witten theory converge (or is asymptotic to) the exact solution due to Witten* [Wit] *and Reshetikhin-Turaev* [RT] *when the parameter k converges to infinity?*

3.3. **Associators and trivalent graphs.** There is also an entirely algebraic approach for the construction of a universal finite type invariant Z_3. The idea is to find some algebraic context within which knot theory is finitely presented — i.e., presented by finitely many generators subject to finitely many relations. If the algebraic context at hand is compatible with the definitions of finite type invariants and of chord diagrams, one may hope to define Z_3 by defining it on the generators in such a way that the relations are satisfied. Thus the problem of defining Z_3 is reduced to finding finitely many elements of $\mathcal{A}$-like spaces which solve certain finitely many equations.

A concrete realization of this idea is in [LM2, BN4] (following ideas from [Dr1, Dr2] on quasitriangular Quasi-Hopf algebras). The relevant "algebraic context" is a category with certain extra operations, and within it, knot theory is generated by just two elements, the braiding and the re-association . Thus to define Z_3 it is enough to find $R = Z_3(\diagup\!\!\diagdown)$ and "an associator" $\Phi = Z_3(|\diagup|)$ which satisfy certain normalization conditions as well as the pentagon and hexagon equations

$$\Phi^{123}\cdot(1\Delta1)(\Phi)\cdot\Phi^{234} = (\Delta11)(\Phi)\cdot(11\Delta)(\Phi),$$

$$(\Delta1)(R^{\pm}) = \Phi^{123}(R^{\pm})^{23}(\Phi^{-1})^{132}(R^{\pm})^{13}\Phi^{312}.$$

As it turns out the solution for R is easy and nearly canonical. But finding an associator Φ is a lot harder. There is a closed form integral expression Φ^{KZ} due to [Dr1] but one encounters the same not-too-well-understood multiple ζ numbers of Section 3.1. There is a rather complicated iterative procedure for finding an associator [Dr2, BN8], [BN4]. On a computer it had been used to find an associator up to degree 7. There is also closed form associator that works only with the Lie super-algebra $gl(1|1)$ [Lie2]. But it remains an open problem to find a closed form formula for a rational associator (existence by [Dr2, BN8]).

On the positive side we should note that the end result, the invariant Z_3, is independent of the choice of Φ and that $Z_3 = Z_1$ [LM2].

There is an alternative (more symmetric and intrinsicly 3-dimensional, but less well documented) description of the theory of associators in terms of knotted trivalent graphs [BNT], [Th2]. There ought to be a perturbative invariant associated with knotted trivalent graphs in the spirit of Section 3.2 and such an invariant should lead to a simple proof that

$Z_2 = Z_3 = Z_1$. But the $\mathcal{E}(D)$ factors remain untamed in this case.

3.4. **Step by step integration.** The last approach for proving the Fundamental Theorem is the most natural and historically the first. But here it is last because it is yet to lead to an actual proof. A weight system $W : \mathcal{A}^r_m \to \mathbb{Q}$ is an invariant of m-singular knots. We want to show that it is the mth derivative of an invariant V of non-singular knots. It is natural to try to integrate W step by step, first finding an invariant V^{m-1} of $(m-1)$-singular knots whose derivative in the sense of (1) is W, then an invariant V^{m-2} of $(m-2)$-singular knots whose derivative is V^{m-1}, and so on all the way up to an invariant $V^0 = V$ whose mth derivative will then be W. If proven, the following conjecture would imply that such an inductive procedure can be made to work:

Conjecture 15. [Hu] *If V^r is a once-integrable invariant of r-singular knots then it is also twice integrable. That is, if there is an invariant V^{r-1} of $(r-1)$-singular knots whose derivative is V^r, then there is an invariant V^{r-2} of $(r-2)$-singular knots whose second derivative is V^r.*

In [Hu] Hutchings reduced this conjecture to a certain appealing topological statement and further to a certain combinatorial-algebraic statement about the vanishing of a certain homology group H^1 which is probably related to Kontsevich's graph homology complex [Ko2] (Kontsevich's H^0 is $\mathcal{A}$, so this is all in the spirit of many deformation theory problems where H^0 enumerates infinitesimal deformations and H^1 is the obstruction to globalization). Hutchings [Hu] was also able to prove the vanishing of H^1 (and hence reprove the Fundamental Theorem) in the simpler case of braids. But no further progress has been made along these lines since [Hu].

4. Some further directions

We would like to touch on a number of significant further directions in the theory of finite type invariants. We will only say a few words on each of those and refer the reader to the literature for further information.

4.1. **The original "Vassiliev" perspective.** V.A. Vassiliev came to the study of finite type knot invariants by studying the infinite dimensional space of all immersions of a circle into $\mathbb{R}^3$ and the topology of the "discriminant", the locus of all singular immersions within the latter space [Va1, Va2]. Vassiliev studied the topology of the complement of the discriminant (the space of embeddings) using a certain spectral sequence and found that certain terms in it correspond to finite type invariants. This later got related to the Goodwillie calculus and back to the configuration spaces of Section 3.2. See [Vol].

4.2. **Interdependent modifications.** The standard definition of finite type invariants is based on modifying a knot by replacing over (or under) crossings with under (or over) crossings. In [Go3] Goussarov generalized this by allowing arbitrary modifications done to a knot — just take any segment of the knot and move it anywhere else in space. The resulting new "finite type" theory turns out to be equivalent to the old one though with a factor of 2 applied to the grading (so an "old" type m invariant is a "new" type $2m$ invariant and vice versa). See also [BN10, Co].

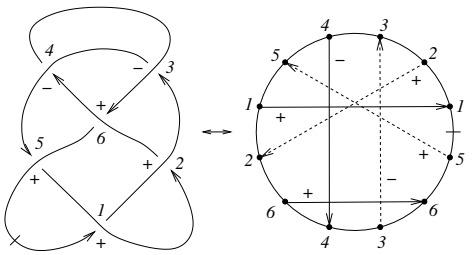
4.3. **n-equivalence, commutators and claspers.** While little is known about the overall power of finite type invariants, much is known about the power of type n invariants for any given n. Goussarov [Go2] defined the notion of n-equivalence: two knots are said to be "n-equivalent" if all their type n invariants are the same. This equivalence relation is well understood both in terms of commutator subgroups of the pure braid group [Sta, NS] and in terms of Habiro's calculus of surgery over "claspers" [Ha] (the latter calculus also gives a topological explanation for the appearance of Jacobi diagrams as in Section 2.5). In particular, already Goussarov [Go2] shows that the set of equivalence classes of knots modulo n-equivalence is a finitely generated Abelian group G_n under the operation of connected sum, and the rank of that group is equal to the dimension of the space of type n invariants.

Ng [Ng] has shown that ribbon knots generate an index 2 subgroup of G_n.

4.4. **Polynomiality and Gauss sums.** Goussarov [Go4] (see also [GPV]) found an intriguing way to compute finite type invariants from a Gauss diagram presentation of a knot, showing in particular that finite type invariants grow as polynomials in the number of crossings n and can be computed in polynomial time in n (though actual computer programs are still missing!).

Gauss diagrams are obtained from knot diagrams in much of the same way as Chord diagrams are obtained from singular knots, except all crossings are counted and not just the double points, and certain over/under and sign information is associated with each crossing/chord so that the knot diagram can be recovered from its Gauss diagram. In the example

below, we also dashed a subdiagram of the Gauss diagram equivalent to the chord diagram :



If G is a Gauss diagram and D is a chord diagram we let $\langle D, G\rangle$ be the number of subdiagrams of G equivalent to D, counted with appropriate signs (to be precise, we also need to base the diagrams involved and count subdiagrams that respect the basing).

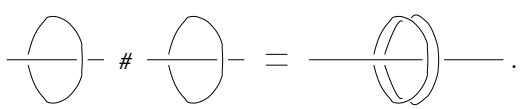
Theorem 16. [Go4, GPV] *If V is a type m invariant then there are finitely many (based) chord diagrams D_i with at most m chords and rational numbers α_i so that $V(K) = \sum_i \alpha_i \langle D_i, G\rangle$ whenever G is a Gauss diagram representing a knot K.*

4.5. **Computing the Kontsevich Integral.** While the Kontsevich Integral Z_1 is a cornerstone of the theory of finite type invariants, it has been computed for surprisingly few knots. Even for the unknot the result is non-trivial:

Theorem 17. *("Wheels",* [BGRT1, BLT]*) The framed Kontsevich integral of the unknot, $Z_1^F(\bigcirc)$, expressed in terms of diagrams in $\hat{\mathcal{B}}$, is given by $\Omega = \exp_{\cup} \sum_{n=1}^{\infty} b_{2n}\omega_{2n}$, where the 'modified Bernoulli numbers' b_{2n} are defined by the power series expansion $\sum_{n=0}^{\infty} b_{2n}x^{2n} = \frac{1}{2}\log\frac{\sinh x/2}{x/2}$, the '$2n$-wheel' ω_{2n} is the free Jacobi diagram made of a $2n$-gon with $2n$ legs (so e.g., $\omega_6 =$* *) and where $\exp_{\cup}$ means 'exponential in the disjoint union sense'.*

Closed form formulas have also been given for the Kontsevich integral of framed unknots, the Hopf link and Hopf chains [BNL] and for torus knots [Ma].

Theorem 17 has a companion that utilizes the same element Ω, the "wheeling" theorem [BGRT1, BLT]. The wheeling theorem "upgrades" the vector space isomorphism $\chi : \mathcal{B} \to \mathcal{A}$ to an algebra isomorphism and is related to the Duflo isomorphism of the theory of Lie algebras. It is amusing to note that the wheeling theorem (and hence Duflo's theorem in the metrized case) follows using finite type techniques from the "$1 + 1 = 2$ on an abacus" identity



4.6. **Taming the Kontsevich Integral.** While explicit calculations are rare, there is a nice structure theorem for the values of the Kontsevich integral, saying that for a knot K and up to any fixed number of loops in the Jacobi diagrams, $\chi^{-1}Z_1(K)$ can be described by finitely many rational functions (with denominators powers of the Alexander polynomial) which dictate the placement of the legs. This structure theorem was conjectured in [Ro], proven in [Kr] and partially generalized to links in [GK].

4.7. **The Rozansky-Witten theory.** One way to construct linear functionals on $\mathcal{A}$ (and hence finite type invariants) is using Lie algebras and representations as in Section 2.5; much of our insight about $\mathcal{A}$ comes this way. But there is another construction for such functionals (and hence invariants), due to Rozansky and Witten [RozWit], using contractions of curvature tensors on hyper-Kähler manifolds. Very little is known about the Rozansky-Witten approach; in particular, it is not known if it is stronger or weaker than the Lie algebraic approach. For an application of the Rozansky-Witten theory back to hyper-Kähler geometry check [HS] and for a unification of the Rozansky-Witten approach with the Lie algebraic approach (albeit at a categorical level) check [RobWil].

4.8. **The Melvin-Morton conjecture and the volume conjecture.** The Melvin-Morton conjecture (stated [MeMo], proven [BNG]) says that the Alexander polynomial can be read off certain coefficients of the coloured Jones polynomial. The Kashaev-Murakami-Murakami volume conjecture (stated [Ka, MuMu], unproven) says that a certain asymptotic growth rate of the coloured Jones polynomial is the hyperbolic volume of the knot complement.

Both conjectures are not directly about finite type invariants but both have ramifications to the theory of finite type invariants. The Melvin-Morton conjecture was first proven using finite type invariants and several later proofs and generalizations (see [BN6]) also involve finite type invariants. The volume conjecture would imply, in particular, that the hyperbolic volume of a knot complement can be read from that knot's finite type invariants, and hence finite type invariants would be at least as strong as the volume invariant.

A particularly noteworthy result and direction for further research is Gukov's [Gu] recent unification of these two conjectures under the Chern-Simons umbrella (along with some relations to three dimensional quantum gravity).

5. Beyond knots

For the lack of space we have restricted ourselves here to a discussion of finite type invariants of knots. But the basic "differentiation" idea of Section 1 calls for generalization, and indeed it has been generalized extensively. We will only make a few quick comments.

Finite type invariants of homotopy links (links where each component is allowed to move across itself freely) and of braids are extremely well behaved. They separate, they all come from Lie algebraic constructions [BN3, Lin2, BN7] and in the case of braids, step by step integration as in Section 3.4 works [Hu] (for homotopy links the issue was not studied).

Finite type invariants of 3-manifolds and especially of integral and rational homology spheres have been studied extensively and the picture is nearly a complete parallel of the picture for knots. There are several competing definitions of finite type invariants (due to [Oh] and then [Go5, Ga, GGP] and more), and they all agree up to regrading. There are weight systems and they are linear functionals on a space $\mathcal{A}(\emptyset)$ which is a close cousin of $\mathcal{A}$ and $\mathcal{B}$ and is related to Lie algebras and hyper-Kähler manifolds in a similar way. There is a notion of a "universal" invariant, and there are several constructions (due to [LMO, Le] and then [BGRT2, KT]), they all agree or are conjectured to agree, and they are related to the Chern-Simons-Witten theory.

Finite type invariants were studied for several other types of topological objects, including knots within other manifolds, higher dimensional knots, virtual knots, plane curves and doodles and more. See [BN6].

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