# FAST KHOVANOV HOMOLOGY COMPUTATIONS

DROR BAR-NATAN

*To Lou Kauffman, who gave us* $\times \mapsto A\,)(\, + A^{-1}\asymp$.

Abstract. We introduce a *local* algorithm for Khovanov Homology computations — that is, we explain how it is possible to "cancel" terms in the Khovanov complex associated with a ("local") tangle, hence canceling the many associated "global" terms in one swoosh early on. This leads to a dramatic improvement in computational efficiency. Thus our program can rapidly compute certain Khovanov homology groups that otherwise would have taken centuries to evaluate.

## Contents



## 1. Introduction

Image source: [BN3].

The "divide and conquer" approach to computation, as applied to knot theory, goes roughly as follows. Suppose a certain knot invariant takes an exponential amount of time to compute, so on a knot with $n$ crossings ($n = 14$ on the right, and it's really a link), it takes roughly $C_1^n$ operations, where $C_1$ is some constant. Suppose also that that same knot invariant can also be computed "in halves"; i.e., it makes sense to compute a "partial" invariant of the left half of the knot, consisting of just $n/2$ crossings, and likewise for the right half of the knot.

*Date*: First edition: Jun. 13, 2006. This edition: May. 20, 2007 .

1991 *Mathematics Subject Classification.* 57M25.

*Key words and phrases.* Categorification, Cobordism, Divide and Conquer, Jones Polynomial, Kauffman Bracket, Khovanov, Knot Invariants, Tangles.

This work was partially supported by NSERC grant RGPIN 262178. See also `http://www.math.toronto.edu/~drorbn/papers/FastKh/`, arXiv:math.GT/0606318 and J. Knot Th. and Rami. **16** (2007) 243–255.

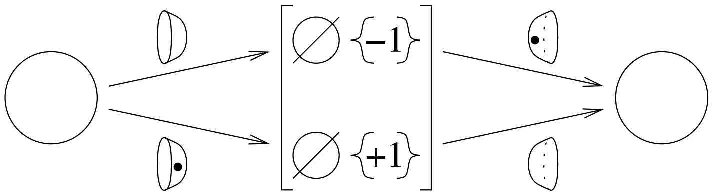



**Figure 1.** Delooping.

Then, with some luck, it takes just $C_1^{n/2}$ operations to do each half and if the assembly of the half-computations into the full one is cheap then the whole computation takes $2C_1^{n/2}$ operations, a lot less than the original $C_1^n$ operations. Of course, by iterating this procedure one can save even more and carry out the computation in $4C_1^{n/4}$ operations, or even just $8C_1^{n/8}$ operations. At the limit the computation time becomes linear in $n$, at least if one ignores the costs of cutting and of assembly.

In reality, cutting is indeed cheap but assembly isn't. Often the invariant of each half knot has to be quite complicated in order to allow for its pairing with every conceivable "other half". This often means that each half-knot invariant must take value in a space whose dimension grows exponentially in the number of strands $b$ that connect the two halves ($b = 4$ in our example). Thus each half computation takes at least $C_2^b$ operations and if luck strikes, it doesn't take much longer. Typically we can expect $b$ to be roughly the "width" of the knot and since we are in the plane, we can expect $b$ to be around $\sqrt{n}$. Thus realistically divide and conquer may reduce $C_1^n$ to $C_2^{\sqrt{n}}$. The latter is still very big, but it is a lot smaller than the former.

In fact, the advantage of "divide and conquer" is so big that it is worthwhile to try this approach even if the assembly cost is more than $C_2^b$ or even if no good estimates for the assembly cost at all exist, as is the case at hand in this paper.

This paper applies the "divide and conquer" approach to the computation of Khovanov homology [Kh, BN1]. We start in Section 3 with a quick review of the local Khovanov theory of [BN2], which amounts to a definition of "Khovanov homology" for half-knots (i.e., for tangles), along with the "horizontal composition" technique necessary for the assembly of the invariants of two tangles into the invariant of their side-by-side composition.

In Section 4 we introduce two simple tools, delooping and Gaussian elimination, that allow us to "simplify" the invariants of tangles. These tools are the keys to the whole paper, as they reduce the complexity of the Khovanov complex associated with a tangle and thus allow for much easier horizontal composition assembly. For the impatient reader, delooping and Gaussian elimination are depicted in Figures 1 and 2 here. If you understand these sketches you've understood the whole paper.

In Section 5 we describe our algorithm. As an illustration, in Section 6 we "run" the algorithm on the figure eight knot. Then in Section 7 we describe an even faster variant of the algorithm and in Section 8 we mention the two available implementations and exhibit a sample computation. The final Section 9 quickly explains how the tools used in this paper lead to a completely automated proof of the invariance of Khovanov homology under Reidemeister moves.

$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix}b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix}b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]$$

is isomorphic to the (direct sum) complex

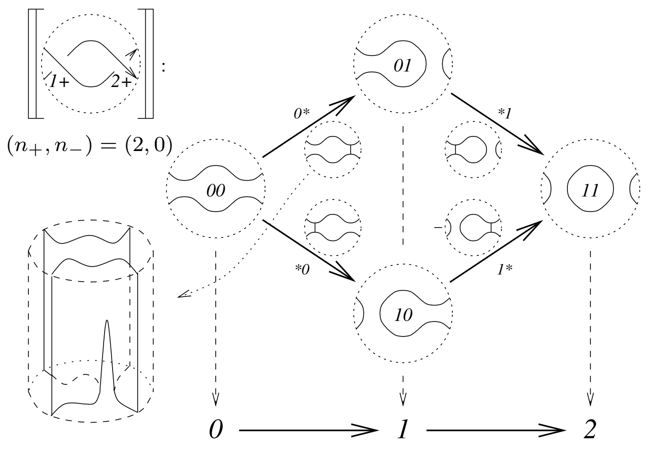
$$[C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix}b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix}b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]$$

**Figure 2.** Gaussian elimination.

## 2. History and Acknowledgement

When I did the research for my paper [BN2] part of my motivation was to set up the framework allowing for "divide and conquer" computations as described in Section 1. But by the time I finished writing [BN2] I completely forgot about that part of my motivation. I wish to thank Marco Mackaay for reminding me! Further thanks to Jeremy Green for implementing the algorithm described here and to Louis Leung, Scott Morrison, Gad Naot, Alexander Shumakovitch ($\times 2$), Adam Sikora and Paul Turner for some comments and suggestions.

## 3. A quick review of the local Khovanov theory

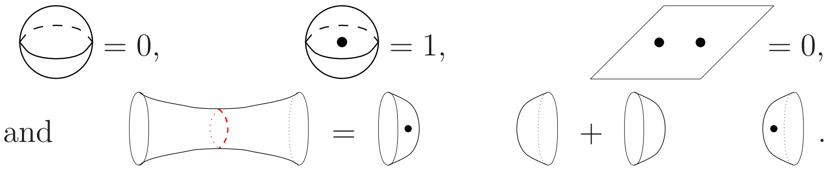



Let us briefly recall the definition of the Khovanov complex for tangles, following [BN2]. Given an $n$-crossing tangle $T$ with boundary $\partial T$, such as the 2-crossing tangle displayed on the right, one constructs an $n$-dimensional "cube" of smoothings and cobordisms between them (as illustrated on the right). This cube is then "flattened" to a "formal complex" $[\![T]\!]$ in the additive category $\mathcal{C}ob^3_{\bullet/l}(\partial T)$ whose objects are formally graded smoothings with boundary $\partial T$ and whose morphisms are formal linear combinations of cobordisms whose tops and bottoms are smoothings and whose side boundaries are $I \times \partial T$, modulo some local relations.

The Khovanov complex $Kh(T)$ of $T$ is obtained from $[\![T]\!]$ by some minor further degree and height shifts depending only of the numbers $n_\pm$ of over- and under-crossings in $T$ (see [BN2, Definition 6.4]). It is a member in the category $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l}(\partial T)))$ of complexes of formal direct sums of objects in $\mathcal{C}ob^3_{\bullet/l}(\partial T)$ and it is invariant up to (formal) homotopies.

For simplicity we are using as the basis to our story one of the simpler cobordism categories $\mathcal{C}ob^3_{\bullet/l}$ that appear in [BN2], rather than the most general one, $\mathcal{C}ob^3_{/l}$ (a fuller treatment appears in [Na]). It is worthwhile to repeat here the local relations that appear in the definition of $\mathcal{C}ob^3_{\bullet/l}$ (see [BN2, Section 11.2]):

(1)




Also recall from [BN2, Section 5] that $[\![\cdot]\!]$, and hence $Kh(\cdot)$, are planar algebra morphisms. That is, if $T_1$ and $T_2$ are tangles and $D(T_1, T_2)$ denotes one of their side-by-side compositions (a side by side placement of $T_1$ and $T_2$ while joining some of their ends in a certain way prescribed by a planar arc diagram $D$), then $[\![D(T_1, T_2)]\!] = D([\![T_1]\!], [\![T_2]\!])$. Here as in [BN2, Section 5] $D([\![T_1]\!], [\![T_2]\!])$ is the "tensor product" operation induced on formal complexes by the horizontal composition operation $D$ on the canopoly $\mathcal{C}ob^3_{\bullet/l}$. In exactly the same sense we also have that $Kh(D(T_1, T_2)) = D(Kh(T_1), Kh(T_2))$.

Thus Khovanov homology is ready for a divide and conquer computation. It makes sense for half-knots (tangles) and there is a composition rule that takes the invariants of the halves and produces the invariant of the whole. But as it stands there is no advantage (yet) for this computation method. If $T_i$ (for $i = 1, 2$) has $n_i$ crossings, the Khovanov cube for $T_i$ consists of $2^{n_i}$ vertices, thus $[\![T_i]\!]$ involves $2^{n_i}$ objects and thus $D([\![T_1]\!], [\![T_2]\!])$ involves $2^{n_1} \cdot 2^{n_2} = 2^{n_1+n_2}$ objects, exactly as many as in $[\![D(T_1, T_2)]\!]$, and nothing has been saved.

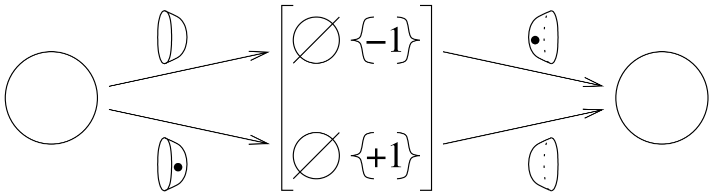
## 4. The tools: delooping and Gaussian elimination

To overcome the difficulty from the previous paragraph we need to learn how to simplify $[\![T_1]\!]$ and $[\![T_2]\!]$ (modulo homotopy) *before* taking their tensor product and thus before the biggest number ($2^{n_1+n_2}$) is encountered. For this we need tools for simplifying complexes over the category $\mathcal{C}ob^3_{\bullet/l}$. These tools are Lemma 4.1 and Lemma 4.2 below.

**Lemma 4.1.** *(Delooping) If an object $S$ in $\mathcal{C}ob^3_{\bullet/l}$ contains a closed loop $\ell$, then it is isomorphic (in $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l})$) to the direct sum of two copies $S'\{+1\}$ and $S'\{-1\}$ of $S$ in which $\ell$ is removed, one taken with a degree shift of $+1$ and one with a degree shift of $-1$. Symbolically, this reads $\bigcirc \equiv \emptyset\{+1\} \oplus \emptyset\{-1\}$.*

*Proof.* Here are the isomorphisms:

(2)




It is easy to verify using (all!) the relations in (1) that the two possible compositions of the morphisms above are both equal to the identity morphisms of the relevant objects. (This is the only place in this paper where the relations in (1) are used). □

**Lemma 4.2.** *(Gaussian elimination, made abstract) If $\phi : b_1 \to b_2$ is an isomorphism (in some additive category $\mathcal{C}$), then the four term complex segment in* $\mathrm{Mat}(\mathcal{C})$

$$\cdots\ [C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix}b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix}b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]\ \cdots \tag{3}$$

*is isomorphic to the (direct sum) complex segment*

$$\cdots\ [C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix}b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix}b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]\ \cdots. \tag{4}$$

*Both these complexes are homotopy equivalent to the (simpler) complex segment*

$$\cdots\ [C] \xrightarrow{(\beta)} [D] \xrightarrow{\left(\epsilon-\gamma\phi^{-1}\delta\right)} [E] \xrightarrow{(\nu)} [F]\ \cdots. \tag{5}$$

*Here $C$, $D$, $E$ and $F$ are arbitrary columns of objects in $\mathcal{C}$ and all Greek letters (other than $\phi$) represent arbitrary matrices of morphisms in $\mathcal{C}$ (having the appropriate dimensions, domains and ranges); all matrices appearing in these complexes are block-matrices with blocks as specified. $b_1$ and $b_2$ are billed here as individual objects of $\mathcal{C}$, but they can equally well be taken to be columns of objects provided (the morphism matrix) $\phi$ remains invertible.*

*Proof.* The two $2 \times 2$ (block) matrices in (3) and (4) differ by invertible row and column operations (i.e., by a "change of basis"). When the corresponding column and row operations are performed on $\begin{pmatrix}\mu & \nu\end{pmatrix}$ and on $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, the results are $\begin{pmatrix}\mu-\nu\gamma\phi^{-1} & \nu\end{pmatrix} = \begin{pmatrix}0 & \nu\end{pmatrix}$ and $\begin{pmatrix}\alpha-\phi^{-1}\delta\beta\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ \beta\end{pmatrix}$ respectively (note that $\mu\phi - \nu\gamma = 0$ and $\phi\alpha - \delta\beta = 0$ as in the original complex the differential squares to 0). Hence the complexes in (3) and (4) are isomorphic.

The complexes in (4) and (5) differ by the removal of a contractible direct summand $0 \longrightarrow b_1 \xrightarrow{\phi} b_2 \longrightarrow 0$ (remember that $\phi$ is invertible). Hence they are homotopy equivalent. $\square$

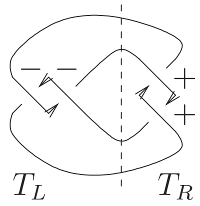
## 5. The algorithm

We thus have a procedure for simplifying complexes $\Omega$ with objects in $\mathcal{C}ob_{\bullet/l}$ (and thus, along with "divide and conquer", we have a potentially fast way of computing Khovanov homologies):

- Whenever an object in $\Omega$ contains a closed loop, double it up using Lemma 4.1, removing the closed loop and inserting $\pm 1$ degree shifts. This done, $\Omega$ becomes $\Omega'$.
- $\Omega'$ is "bigger" than $\Omega$, but it is made up of many fewer possible objects. Thus it is likely that many morphisms in $\Omega'$ are isomorphisms. Whenever you find one, cancel it using Lemma 4.2. Call the result of doing this iteratively $\Omega''$.

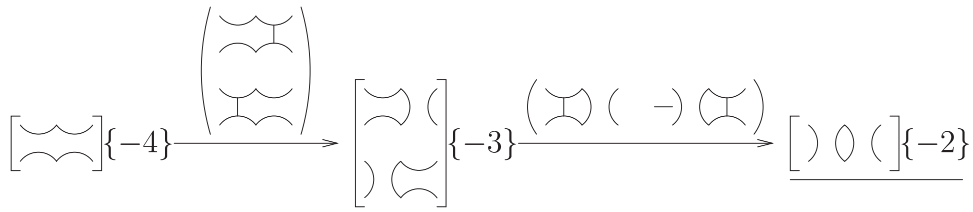
A priori, we cannot guarantee that $\Omega''$ will be simpler than $\Omega$. But experimentation shows that it is, as seen in the next few sections.

## 6. The figure eight knot



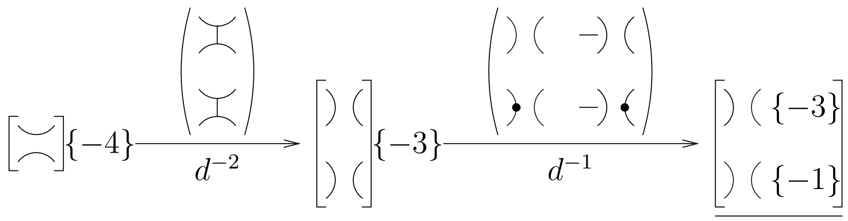

Our first example is the figure eight knot, cut in half into two tangles $T_L$ and $T_R$ as shown on the right. $T_L$ is the tangle ⋌⋌. Its Khovanov complex is the complex $\Omega_1 = Kh(⋌⋌)$ appearing below. Here and later we use the same notational conventions as in [BN2]. Thus the height zero object in the complex is underlined, and ⌶ is the saddle cobordism whose domain is ≍ and whose range is )(:

$\Omega_1$ :




Only one of the objects in $\Omega_1$ contains a loop — the last one. Delooping it using Lemma 4.1 we get the complex $\Omega_2$, which is isomorphic to $\Omega_1$ (we've also replaced some of the smoothings and cobordisms appearing in $\Omega_1$ with isotopic ones):

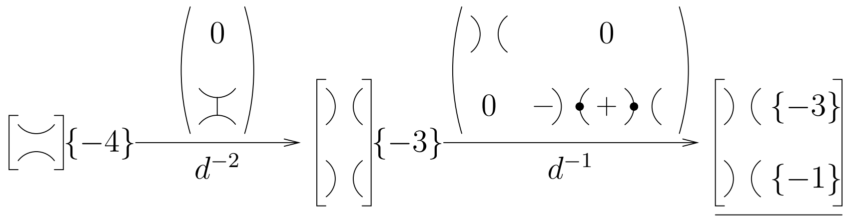
$\Omega_2$ :




The symbol )( here, when appearing as a cobordism, denotes the identity automorphism of the smoothing )(. Likewise ⦒( and )⦑ denote that same cobordism, with an extra dot on the left (or right) "curtain".

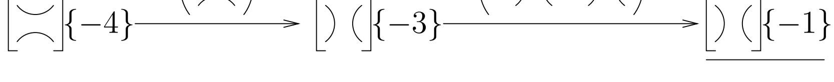
The upper left entry in the differential $d^{-1}$ in $\Omega_2$ is an isomorphism (so is the upper right entry, but one is enough). So we can apply the first part of Lemma 4.2 taking $\phi$ to be that upper left entry. The result is the complex $\Omega_3$ below, which is isomorphic to $\Omega_2$ and $\Omega_1$. Note that in this case we have no "$\begin{pmatrix}\mu & \nu\end{pmatrix}$" term, and that in this case, $-\gamma\phi^{-1}\delta = -⦒( \circ )( \circ (-)() = +⦒($:

$\Omega_3$ :




Dropping the contractible summand as in the second part of Lemma 4.2 we get the complex $\Omega_4$:

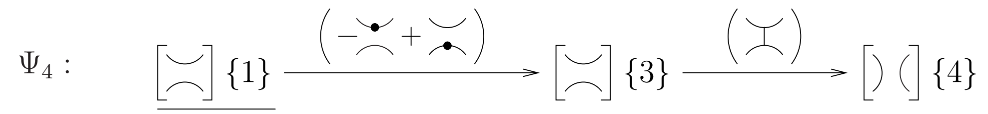
$\Omega_4$ :



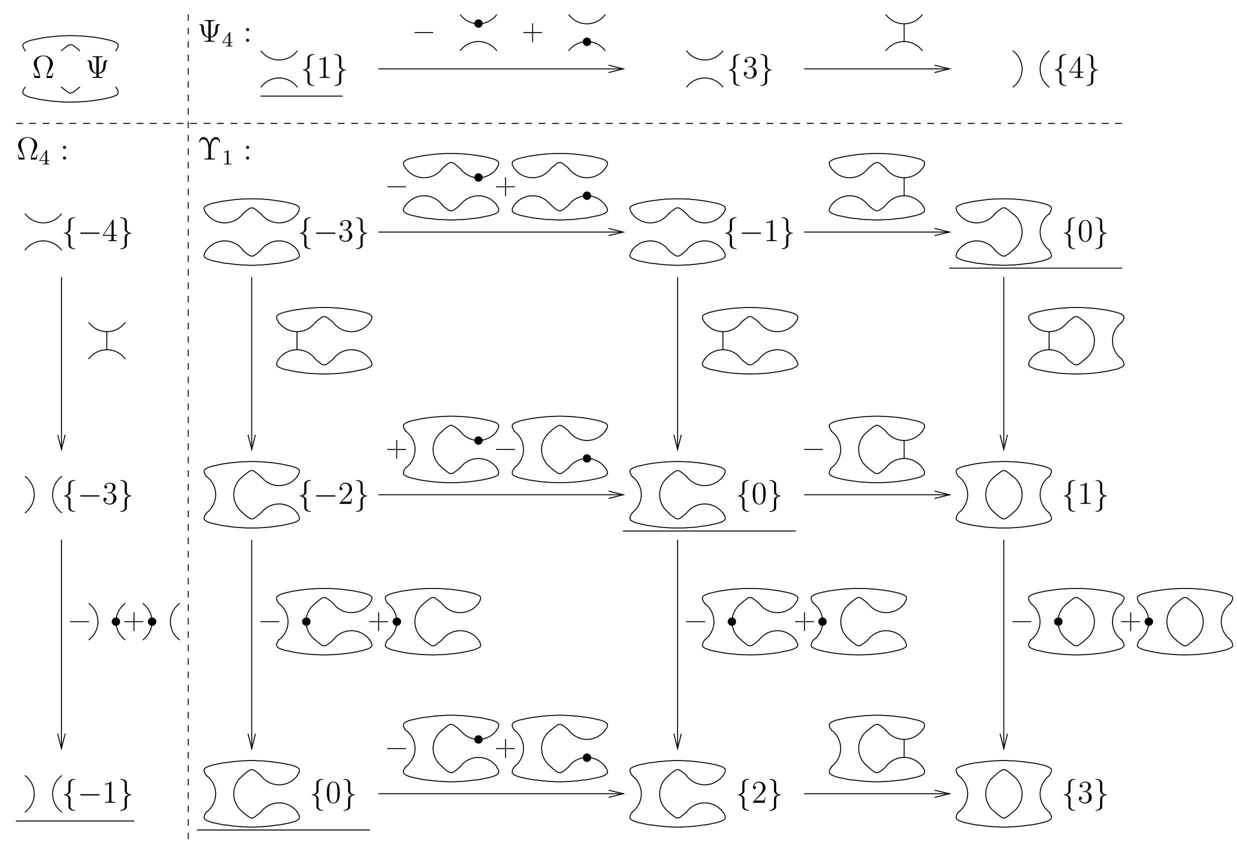
The complex $\Omega_4$ contains fewer objects than $\Omega_1$, hence from a computational perspective it is indeed simpler. The savings, about 25%, may not appear to be much, but remember —

- This 25% will be compounded with whatever savings we will incur later in the computation.
- This is just a sample. In real life the savings are greater.

In a similar manner we can compute and simplify the Khovanov complex of the right half of the Figure Eight knot, the tangle $T_R$. The result is the complex $\Psi_4$:

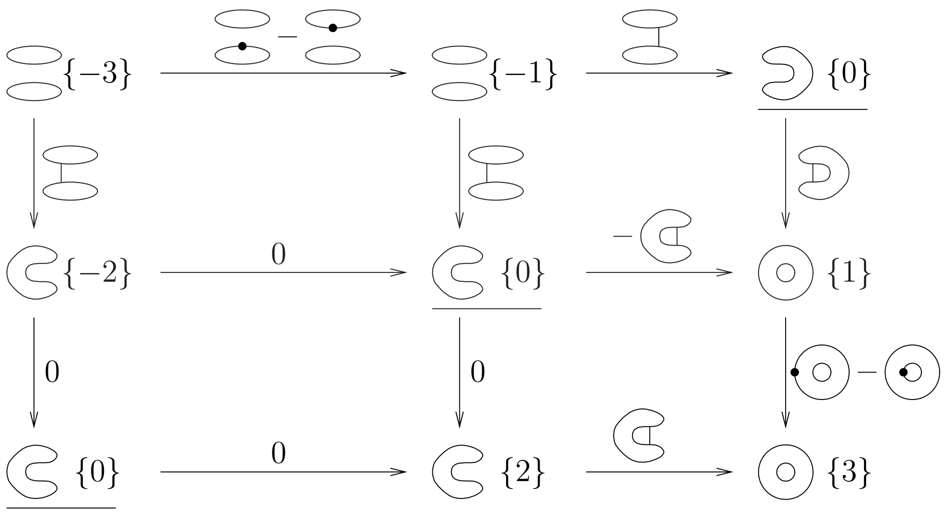



Next we have to take the "tensor product" of $\Omega_4$ with $\Psi_4$ using the same side-by-side composition used to make the Figure Eight knot out of $T_L$ and $T_R$. The result is the double complex $\Upsilon_1$ below. To prevent clutter we've done away with the vector and matrix brackets; also, notice the signs in the middle row, sprinkled as commonly practiced on tensor products.




$\Upsilon_1$ may seem monstrous, but yet, it contains just 9 objects, as compared to the 16 objects one sees in a direct computation of the Khovanov complex of a 4-crossing knot. Anyway, it is best to re-write $\Upsilon_1$ a bit before proceeding. The smoothings and cobordisms can be smoothed out, and dots can be moved around cobordisms so as to cancel the four differences on the lower left of the $\Upsilon_1$ diagram. The result is $\Upsilon_2$ below.

$\Upsilon_2$ :




If we were a computer program we would have now "flattened" the double complex $\Upsilon_2$ to a single complex of the schematic form $(\cdot) \to (:) \to (\vdots) \to (:) \to (\cdot)$. Such "single" complexes are more easily manipulated on a computer. But it is unlikely this paper will ever be appreciated by anything but humans. So we'll stick to the more readable double complex form while remembering that we really have just one differential going south and east.

Anyway, the next step is to replace every loop in every object in $\Upsilon_2$ with a pair of (degree-shifted) empty sets, as in Lemma 4.1, while replacing the differentials with their compositions with the explicit isomorphisms of (2). But there's nothing but loops in $\Upsilon_2$, so we are left with a complex $\Upsilon_3$ in which all the objects are degree-shifted empty sets and all the morphisms are (matrices of) scalar multiples of the empty cobordism (note that modulo the relations in (1) all closed surfaces reduce to scalars):

$$\Upsilon_3: \quad \begin{tikzcd}[column sep=huge, row sep=huge]
\begin{bmatrix}\emptyset\{-5\}\\ \emptyset\{-3\}\\ \emptyset\{-3\}\\ \emptyset\{-1\}\end{bmatrix} \arrow[r, "\begin{pmatrix}0&1&-1&0\\0&0&0&-1\\0&0&0&1\\0&0&0&0\end{pmatrix}"] \arrow[d, "m\begin{pmatrix}0&1&1&0\\0&0&0&1\end{pmatrix}"] & \begin{bmatrix}\emptyset\{-3\}\\ \emptyset\{-1\}\\ \emptyset\{-1\}\\ \emptyset\{1\}\end{bmatrix} \arrow[r, "\begin{pmatrix}0&1&1&0\\0&0&0&1\end{pmatrix}", "m"'] \arrow[d, "m\begin{pmatrix}0&1&1&0\\0&0&0&1\end{pmatrix}"] & \underline{\begin{bmatrix}\emptyset\{-1\}\\ \emptyset\{1\}\end{bmatrix}} \arrow[d, "\Delta\begin{pmatrix}1&0\\0&1\\0&1\\0&0\end{pmatrix}"] \\
\begin{bmatrix}\emptyset\{-3\}\\ \emptyset\{-1\}\end{bmatrix} \arrow[r, "0"] \arrow[d, "0"] & \underline{\begin{bmatrix}\emptyset\{-1\}\\ \emptyset\{1\}\end{bmatrix}} \arrow[r, "\begin{pmatrix}-1&0\\0&-1\\0&-1\\0&0\end{pmatrix}", "-\Delta"'] \arrow[d, "0"] & \begin{bmatrix}\emptyset\{-1\}\\ \emptyset\{1\}\\ \emptyset\{1\}\\ \emptyset\{3\}\end{bmatrix} \arrow[d, "\begin{pmatrix}0&1&-1&0\\0&0&0&-1\\0&0&0&1\\0&0&0&0\end{pmatrix}"] \\
\underline{\begin{bmatrix}\emptyset\{-1\}\\ \emptyset\{1\}\end{bmatrix}} \arrow[r, "0"] & \begin{bmatrix}\emptyset\{1\}\\ \emptyset\{3\}\end{bmatrix} \arrow[r, "\begin{pmatrix}1&0\\0&1\\0&1\\0&0\end{pmatrix}", "\Delta"'] & \begin{bmatrix}\emptyset\{1\}\\ \emptyset\{3\}\\ \emptyset\{3\}\\ \emptyset\{5\}\end{bmatrix}
\end{tikzcd}$$

Note in passing that the matrices $m$ and $\Delta$ appearing here are the matrices representing the product and the co-product of [Kh] relative to the basis $(X, 1)$ used there (or the basis $(v_-, v_+)$ used in [BN1]). This is essentially the content of [BN2, Section 9.1].

We can now apply Lemma 4.2 repeatedly to $\Upsilon_3$, until no invertible entries remain in any of the matrices. Over $\mathbb{Q}$ any non-zero number is invertible so our process stops when all matrices are 0. Thus a lengthy iterated application of Lemma 4.2 stops at the complex $\Upsilon_4$ shown on the right (a human could save quite a lot by being clever, but that's not our point here).

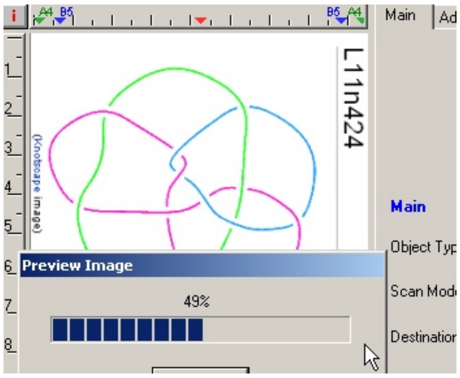
$$\begin{tikzcd}
[\emptyset\{-5\}] \arrow[r, "0"] \arrow[d, "0"] & [\emptyset\{-1\}] \arrow[r, "0"] \arrow[d, "0"] & \underline{[]} \arrow[d, "0"] \\
[] \arrow[r, "0"] \arrow[d, "0"] & \underline{[]} \arrow[r, "0"] \arrow[d, "0"] & [\emptyset\{1\}] \arrow[d, "0"] \\
\underline{\begin{bmatrix}\emptyset\{-1\}\\ \emptyset\{1\}\end{bmatrix}} \arrow[r, "0"] & [] \arrow[r, "0"] & [\emptyset\{5\}]
\end{tikzcd}$$

Belatedly flattening the double complex $\Upsilon_4$ we arrive at our final answer, the complex

$$\Upsilon_5: \qquad [\emptyset\{-5\}] \xrightarrow{0} [\emptyset\{-1\}] \xrightarrow{0} \underline{\begin{bmatrix}\emptyset\{-1\}\\ \emptyset\{1\}\end{bmatrix}} \xrightarrow{0} [\emptyset\{1\}] \xrightarrow{0} [\emptyset\{5\}]\,. \tag{6}$$

To recover homology groups out of $\Upsilon_5$ we need to apply to it some functor $\mathcal{F}$ taking $\mathcal{C}ob^3_{\bullet/l}$ to graded vector spaces, and then take homology. The latter step (computing the homology) is the do nothing operation as all differentials are 0. Typically (e.g., as in [BN2, Section 11.2]) the functor $\mathcal{F}$ maps the empty smoothing to the one dimensional vector space (call it $\mathbb{Q}$) at degree 0. And so we can read directly from (6) that the "conventional" Khovanov homology over $\mathbb{Q}$ of the figure eight knot is 6-dimensional with generators at bidegrees $(-2,-5)$, $(-1,-1)$, $(0,-1)$, $(0,1)$, $(1,1)$ and $(2,5)$.

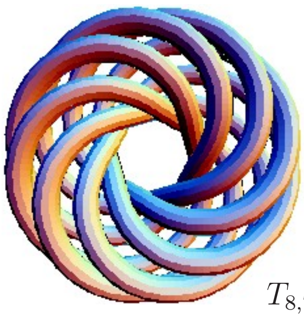
## 7. A faster algorithm




It turns out that there is a somewhat better way to assemble our two main tools into a running algorithm. Instead of computing each "half knot" and combining the result, "scan" the knot from left to right (or top to bottom, as in the picture on the left), adding one crossing at a time. After each crossing is added, use delooping (Lemma 4.1) and Gaussian elimination (Lemma 4.2) to simplify the result.

Since we have no rigorous estimate, we have to settle with a possibly naive estimation of the complexity of the "divide and conquer" algorithm and of the new "scanning" algorithm. In both cases the bottleneck ought to be where the knot is widest; if the width at the widest cut is $W$, we expect the complex corresponding to either half of the knot to be of size $C^W$ for some $C$. In the "divide and conquer" algorithm we need to "tensor multiply" two such complexes so for a little while we have to hold a double complex of total size of about $(C^W)^2 = C^{2W}$ (compare with the complex $\Upsilon_1$ above). In the "scanning" algorithm we only need to tensor multiply the complex for the left half-knot with the complex of a single crossing, whose size is 2. So we only see a double complex of size $2C^W$ before we get the chance to simplify again.

## 8. Computer programs



$T_{8,7}$

Two implementations of the "scanning" algorithm are available, both as a part of the knot theory package `KnotTheory‘` [KT]. The first one, `FastKh`, was written by the author just to test the principle, with no attempt at optimization. Yet for example, it was able to compute the Khovanov homology of the 35-crossing $(7,6)$ torus knot $T_{7,6}$ in about one day of work; a task that would have taken about a $1,000$ years without the use of tangles. The second one, `JavaKh` was written by Jeremy Green, a summer student of the author's at the University of Toronto, in the summer of 2005. The computation leading to Table 1, for example, of the Khovanov homology of the 48-crossing $(8,7)$ torus knot $T_{8,7}$, takes a few minutes using `JavaKh`. The $(r,j)$ entry of that table contains the degree $2r+j$ piece of the $r$th integral Khovanov homology $\mathcal{G}_{2r+j}H^r(T_{8,7})$ of $T_{8,7}$.

It is worthwhile to note a certain curious feature of the integral Khovanov homology of $T_{8,7}$. Recall [Lee, Ra] that for any knot $K$ there is a spectral sequence whose $E^2$ term is $\mathcal{G}_jH^r(K)$, whose odd differentials vanish, and which converges to the associated graded space of the Lee homology of $K$ which other than 2-torsion is supported at $r=0$ (see also [MTV]). Thus something must eliminate the $\mathbb{Z}_7$ in the $(15,31)$ box of Table 1. When presented as in Table 1, the second differential $d_2$ of the spectral sequence goes one box down and one to

| | $j$=23 | $j$=25 | $j$=27 | $j$=29 | $j$=31 | $j$=33 | $j$=35 | $j$=37 | $j$=39 | $j$=41 | $j$=43 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $r$=0 | | | | | | | | | | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $r$=1 | | | | | | | | | | | |
| $r$=2 | | | | | | | | | | $\mathbb{Z}$ | |
| $r$=3 | | | | | | | | | | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| $r$=4 | | | | | | | | | $\mathbb{Z}$ | $\mathbb{Z}$ | |
| $r$=5 | | | | | | | | | | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $r$=6 | | | | | | | | $\mathbb{Z}$ | $\mathbb{Z}$ | | |
| $r$=7 | | | | | | | | $\mathbb{Z}_2$ | $\mathbb{Z}\oplus\mathbb{Z}_2$ | $\mathbb{Z}$ | |
| $r$=8 | | | | | | | $\mathbb{Z}$ | $\mathbb{Z}^2$ | | | |
| $r$=9 | | | | | | | | $\mathbb{Z}\oplus\mathbb{Z}_2$ | $\mathbb{Z}^2$ | | |
| $r$=10 | | | | | | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | | |
| $r$=11 | | | | | | $\mathbb{Z}_2$ | $\mathbb{Z}\oplus\mathbb{Z}_2^2$ | $\mathbb{Z}^3$ | | | |
| $r$=12 | | | | | $\mathbb{Z}$ | $\mathbb{Z}^3$ | $\mathbb{Z}$ | $\mathbb{Z}_2\oplus\mathbb{Z}_5$ | $\mathbb{Z}$ | | |
| $r$=13 | | | | | | $\mathbb{Z}\oplus\mathbb{Z}_2^2$ | $\mathbb{Z}^4\oplus\mathbb{Z}_2$ | $\mathbb{Z}$ | | | |
| $r$=14 | | | | | $\mathbb{Z}^2$ | $\mathbb{Z}\oplus\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}$ | | | |
| $r$=15 | | | | | $\mathbb{Z}_2^2\oplus\mathbb{Z}_7$ | $\mathbb{Z}^4\oplus\mathbb{Z}_2$ | $\mathbb{Z}^2$ | | | | |
| $r$=16 | | | | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}^2\oplus\mathbb{Z}_2$ | $\mathbb{Z}$ | | | |
| $r$=17 | | | | $\mathbb{Z}_2$ | $\mathbb{Z}^3\oplus\mathbb{Z}_2$ | $\mathbb{Z}^3$ | | | | | |
| $r$=18 | | | $\mathbb{Z}$ | $\mathbb{Z}^2\oplus\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}\oplus\mathbb{Z}_4$ | $\mathbb{Z}$ | | | | |
| $r$=19 | | | $\mathbb{Z}_2$ | $\mathbb{Z}^2\oplus\mathbb{Z}_2^2$ | $\mathbb{Z}^3$ | | | | | | |
| $r$=20 | | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}\oplus\mathbb{Z}_2^2$ | $\mathbb{Z}$ | | | | | |
| $r$=21 | | | $\mathbb{Z}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_7$ | $\mathbb{Z}^3\oplus\mathbb{Z}_2$ | $\mathbb{Z}_2$ | | | | | | |
| $r$=22 | | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2\oplus\mathbb{Z}_4\oplus\mathbb{Z}_5\oplus\mathbb{Z}_7$ | $\mathbb{Z}$ | | | | | | |
| $r$=23 | | $\mathbb{Z}_2\oplus\mathbb{Z}_7$ | $\mathbb{Z}^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | | | | | | |
| $r$=24 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2\oplus\mathbb{Z}_5\oplus\mathbb{Z}_7$ | $\mathbb{Z}$ | | | | | | | |
| $r$=25 | | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_5$ | | | | | | | |
| $r$=26 | | | $\mathbb{Z}_2\oplus\mathbb{Z}_3$ | $\mathbb{Z}_2$ | | | | | | | |

**Table 1.** The integral Khovanov homology groups $\mathcal{G}_{2r+j}H^r(T_{8,7})$.

the right. Nothing at boxes $(14, 29)$ or $(16, 33)$ can pair with our $\mathbb{Z}_7$, so it must survive to $E^3$ and hence to $E^4$. The fourth differential $d_4$ goes one box down and three boxes to the right in Table 1. All the higher differentials go even further to the right, so they certainly can't eliminate our $\mathbb{Z}_7$. Thus the only option is that $d_4$ eliminates the $\mathbb{Z}_7$ at $(15, 31)$ and the only way this can happen is if it pairs it with the $\mathbb{Z}$ at $(16, 37)$ which must therefore have been turned into a $\mathbb{Z}_7$ by $d_2$.

Thus there is a non-vanishing higher differential in the Lee-Rasmussen spectral sequence of $T_{8,7}$.

The curious $\mathbb{Z}_7$ in Table 1 was noted by Paul Turner (private communication); I'd like to thank him for noting it and to thank Gad Naot for his help explaining it.

A very similar $\mathbb{Z}_3$, also forcing a non-vanishing $d_4$, was noted in the Khovanov homology of $T_{6,-5}$ about a year earlier by Alexander Shumakovitch (private communication) using the same computer program.

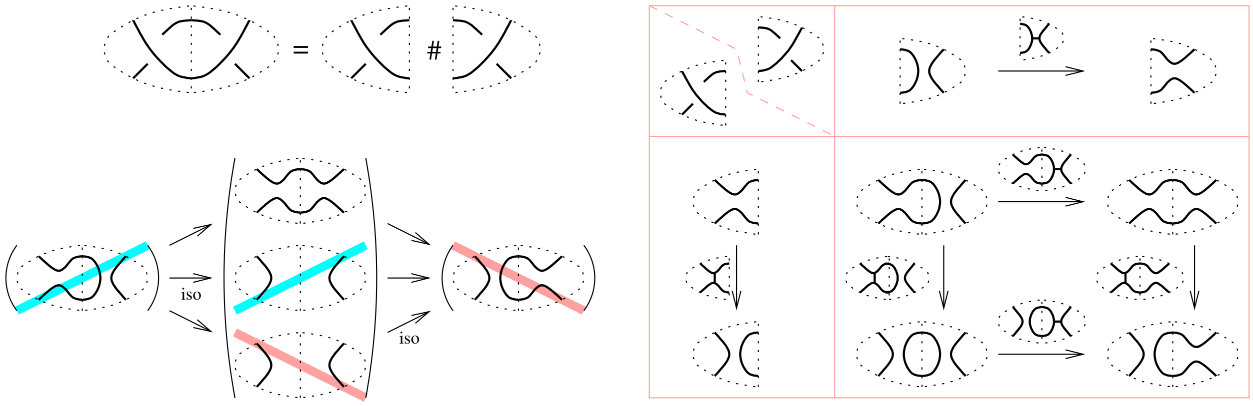



**Figure 3.** Proof of invariance under $R2$.

## 9. The Reidemeister moves

One further advantage of the ability to compute with tangles is that the proof of up-to-homotopy invariance of the Khovanov complex under Reidemeister moves becomes mechanical. All that one has to do is to compute and simplify the complexes corresponding to each side of each Reidemeister move. One gets the same result for both sides of any given Reidemeister move and hence invariance is proven[1].

As an example, Figure 3 proves invariance under the second Reidemeister move $R2$ in summary form. Start on the upper left, where the "hard side" of $R2$ is presented as the side-by-side product of two single-crossing tangles. Move on to the right where the corresponding double complex is shown, and back to the bottom left where that complex is flattened, delooped and the two isomorphisms that appear in the result are canceled out. What remains is a single-entry complex equal to the "easy side" of $R2$. For simplicity degree shifts are ignored in Figure 3.

## References


[BN1] D. Bar-Natan, *On Khovanov's categorification of the Jones polynomial,* Algebraic and Geometric Topology **2-16** (2002) 337–370, `http://www.math.toronto.edu/~drorbn/papers/Categorification/`, arXiv:math.GT/0201043.

[BN2] D. Bar-Natan, *Khovanov's Homology for Tangles and Cobordisms,* Geometry and Topology **9-33** (2005) 1443–1499, `http://www.math.toronto.edu/~drorbn/papers/Cobordism/`, arXiv:math.GT/0410495.

[BN3] D. Bar-Natan, picture of Susan Williams' shirt pin at `http://www.math.toronto.edu/~drorbn/Gallery/KnottedObjects/SusanWilliamsShirtPin.html`.

[Kh] M. Khovanov, *A categorification of the Jones polynomial,* Duke Mathematical Journal **101-3** (2000) 359–426, arXiv:math.QA/9908171.

[KT] `KnotThery‘`, a knot theory mathematica package, `http://katlas.math.toronto.edu/wiki/KnotTheory`.

[Lee] E. S. Lee, *An Endomorphism of the Khovanov Invariant,* Advances in Mathematics **197** (2005) 554–586, arXiv:math.GT/0210213.


[1]In fact, the computer programs discussed above can carry out these computations, turning them literally mechanical. Though the presently available front ends for those programs are only configured to take knots as inputs.

[MTV] M. Mackaay, Paul Turner and P. Vaz, *A remark on Rasmussen's invariant of knots,* arXiv:math.GT/0509692.

[Na] G. Naot, *On the Algebraic Structure of Bar-Natan's Universal Geometric Complex and the Geometric Structure of Khovanov Link Homology Theories,* arXiv:math.GT/0603347.

[Ra] J. A. Rasmussen, *Khovanov homology and the slice genus,* arXiv:math.GT/0402131.

[Wo] S. Wolfram, *The Mathematica Book,* Cambridge University Press, 1999 and `http://www.wolfram.com`.



Department of Mathematics, University of Toronto, Toronto Ontario M5S 2E4, Canada

*E-mail address*: `drorbn@math.toronto.edu`

*URL*: `http://www.math.toronto.edu/~drorbn`