SOME COMPUTATIONS RELATED TO VASSILIEV INVARIANTS

DROR BAR-NATAN

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Abstract. This is a regularly updated collection of results of computations related to Vassiliev invariants. It is distributed using photocopiers, laser printers, and the internet. If you have data to add, please let me know!

Contents

As a warmup, let as start with a reproduction of the table in section 6.1 of [1]:

m	0	1	2	3	4	5	6	7	8	9
$\dim \mathcal{G}_m\mathcal{A}$	1	1	2	3	6	10	19	33	60	104
$\dim \mathcal{G}_m\mathcal{W}$	1	0	1	1	3	4	9	14	27	44
$\dim \mathcal{G}_m\mathcal{P}$	1	1	1	1	2	3	5	8	12	18
$\dim \mathcal{G}_m Lie$	1	1	2	3	6	10	19	33	60	104
diagrams	1	0	1	2	7	36	300	$3,218$	$42,335$	$644,808$
relations	0	0	0	2	15	144	$1,645$	$21,930$	$334,908$	$5,056,798$

The latest edition of this collection of results is available at `http://www.ma.huji.ac.il/~drorbn` and at `file://ftp.ma.huji.ac.il/drorbn`.

Remark 1.

- $\mathcal{G}_m$ obj denotes the degree m piece of a graded object obj. $\mathcal{A}$ is the space of chord diagrams modulo the $4T$ relation (see e.g. [1]). $\mathcal{W}$ is the space of weight systems, that is, functionals $\mathcal{A} \to \mathbf{Q}$ that vanish on chord diagrams having an isolated chord. $\mathcal{P}$ is the space of primitive elements of $\mathcal{A}$ (recall that $\mathcal{A}$ is a commutative and co-commutative Hopf algebra). $\mathcal{G}_m Lie$ is the subspace of $(\mathcal{G}_m\mathcal{A})^*$ spanned by the Lie algebraic weight systems of [1]. In fact, up to degree 9, it is spanned by the weight systems corresponding to classical groups.
- Vogel [10] proved that $\dim \mathcal{G}_m SSLie < \dim \mathcal{G}_m\mathcal{A}$ for large enough values of m, where $SSLie$ denotes the class of semi-simple Lie and super-Lie algebras.
- The numbers in the last two rows indicate the size of the matrices that had to be row reduced in order to compute $\dim \mathcal{G}_m\mathcal{W}$. These numbers have no real significance — they somewhat depend on the details of the algorithm chosen — and are displayed only so as to give an impression of the complexity of the problem.
- The problem is highly exponential and it is unlikely that it will be possible to use the same techniques to compute $\dim \mathcal{G}_{10}\mathcal{A}$.
- It took about $2,106$ lines of C++ code and about 10 days of CPU time to compute the above numbers. The programs are available at my ftp/http sites.

1. Knots

The following is a table of the dimensions $\beta_{m,u}$ and $\varphi_{m,u}$ of $\mathcal{G}_m\mathcal{B}_u$ and of im $\Phi|_{\mathcal{G}_m\mathcal{B}_u}$ where $\mathcal{G}_m\mathcal{B}_u$ is the space of connected Chinese Characters of degree m having exactly u univalent vertices, divided by the AS and IHX relations. This space as well as the map $\Phi : \mathcal{B}_u \to \{\text{marked surfaces}\}$ are defined in [1].

		$u=0$	$u=2$	$u=4$	$u=6$	$u=8$	$u=10$	total ($u>0$)
$m=1$	β	1	1					1
	φ	1	1					1
$m=2$	β	1	1					1
	φ	1	1					1
$m=3$	β	1	1					1
	φ	1	1					1
$m=4$	β	2	1	1				2
	φ	2	1	1				2
$m=5$	β	2	2	1				3
	φ	2	2	1				3
$m=6$	β	3	2	2	1			5
	φ	3	2	2	1			5
$m=7$	β	4	3	3	2			8
	φ	4 (3)	3	3	2			8
$m=8$	β	5	4	4	3	1		12
	φ	5 (4)	4 (3)	4	3	1		12 (11)
$m=9$	β	?	5	6	5	2		18
	φ	≥ 6 (4)	5 (4)	6 (5)	5	2		18 (16)
$m=10$	β	?	?	?	?	4	1	?
	φ	≥ 8 (5)	≥ 6 (4)	≥ 8 (6)	≥ 8 (7)	4	1	≥ 27 (22)
$m=11$	β	?	?	?	?	8	2	?
	φ	?	≥ 8 (5)	≥ 10 (7)	≥ 11 (9)	8 (7)	2	≥ 39 (30)

Remark 2.
- Up to and including $m=9$, $\beta_{m,u}=0$ if u is odd. For $m=10$ this is not known yet. $\varphi_{m,u}=0$ if u is odd for all m.
- When different than $\varphi_{m,u}$, the dimension of im Φ projected to orientable surfaces appears in parenthesis after $\varphi_{m,u}$.
- As was noticed by M. Kontsevich [6], for every m one has $\beta_{m,0}=\beta_{m+1,2}$, $\beta_{m,1}=0$, and similarly for φ.
- Vogel [10] proved that $\varphi_{m,u}<\beta_{m,u}$ for large enough values of m and for $u=0,2$. The same is expected for $u>2$ as well.
- The symbol '$\geq$' appearing in some of the entries above can most likely be replaced by '$=$'. $\varphi_{m,u}$ was computed by computing Φ on a large number of random Chinese characters, but not on all of them. So there is a (small) probability that something in the image of Φ was missed.
- The programs used in the computation of the above numbers are available from my ftp/http sites.

Similarly, here are some values of $\varphi_{m,u}$ for higher m and u. Notice that most of these numbers are just lower bounds, for the same reason as in remark 2. Due to time constrains, the lowest entry in each column was computed by computing Φ on a relatively small number of random Chinese characters, and the probability of mistake in these cases is higher.

	$u=6$	$u=8$	$u=10$	$u=12$	$u=14$	$u=16$	$u=18$
$m=12$	$\geq 14\,(10)$	$\geq 12\,(10)$	5	1			
$m=13$	?	$\geq 18\,(13)$	$\geq 10\,(9)$	3			
$m=14$	?	$\geq 22\,(14)$	$\geq 17\,(14)$	7	1		
$m=15$	?	?	?	$\geq 15\,(13)$	3		
$m=16$	?	?	?	$\geq 27\,(21)$	8		
$m=17$	?	?	?	?	$\geq 19\,(16)$	3	
$m=18$	?	?	?	?	?	10 (?)	1
$m=19$	?	?	?	?	?	?	4

Remark 3. The dimensions $\beta_{m,u}$ discussed above are the same as the dimensions $\beta_m(u')$ discussed below, when $u'=(u)$ is a partition of length 1. Thus some more $\beta_{m,u}$'s, for odd values of u, can be read from the tables in sections 2.1.10–2.1.20. These additional dimensions all vanish. See also remarks 10 and 11.

2. String Links

Next, let us turn to Vassiliev invariants of string links.

2.1. Chinese Characters. By [2], the associated graded space to the filtered space of Vassiliev invariants of string links is the dual of $\mathcal{A}^{sl}$, which is isomorphic to $\mathcal{B}^{sl}$, where $\mathcal{B}^{sl}$ is the space of Chinese characters with 'colored' univalent vertices (modulo IHX and AS, of course). Thus $\mathcal{B}^{sl}$ has many different gradings — by the total degree m (= half the number of vertices), and by the number of univalent vertices of each color. For any partition $u=(u_1\geq u_2\geq\cdots\geq u_l)$, let $\beta_m(u)$ be the dimension of the space $\mathcal{B}_m(u)$ of *connected* Chinese characters of total degree m and of u_i univalent vertices of color υ_i for each $1\leq i\leq l$ (modulo IHX and AS), where $\Upsilon=\{\upsilon_i\}$ is some set of colors. For simple counting reasons, it is clear that $\beta_m(u)=0$ unless $|u|\leq m+1$, where $|u|=\sum u_i$. The following are lists of the dimensions $\beta_m(u)$ presently known to me:

2.1.1. *Degree 1.*

$$\beta_1()=1 \qquad \beta_1(1)=0 \qquad \beta_1(2)=1 \qquad \beta_1(11)=1$$

2.1.2. *Degree 2.*

$$\beta_2()=1 \qquad \beta_2(2)=1 \qquad \beta_2(3)=0 \qquad \beta_2(111)=1$$

$$\beta_2(1)=0 \qquad \beta_2(11)=1 \qquad \beta_2(21)=0$$

2.1.3. *Degree 3.*

$\beta_3() = 1$	$\beta_3(11) = 1$	$\beta_3(111) = 1$	$\beta_3(22) = 1$
$\beta_3(1) = 0$	$\beta_3(3) = 0$	$\beta_3(4) = 0$	$\beta_3(211) = 1$
$\beta_3(2) = 1$	$\beta_3(21) = 0$	$\beta_3(31) = 0$	$\beta_3(1111) = 2$

2.1.4. *Degree 4.*

$\beta_4() = 2$	$\beta_4(21) = 0$	$\beta_4(211) = 2$	$\beta_4(311) = 1$
$\beta_4(1) = 0$	$\beta_4(111) = 1$	$\beta_4(1111) = 3$	$\beta_4(221) = 1$
$\beta_4(2) = 1$	$\beta_4(4) = 1$	$\beta_4(5) = 0$	$\beta_4(2111) = 3$
$\beta_4(11) = 1$	$\beta_4(31) = 1$	$\beta_4(41) = 0$	$\beta_4(11111) = 6$
$\beta_4(3) = 0$	$\beta_4(22) = 2$	$\beta_4(32) = 0$	

2.1.5. *Degree 5.*

$\beta_5() = 2$	$\beta_5(31) = 1$	$\beta_5(221) = 2$	$\beta_5(321) = 2$
$\beta_5(1) = 0$	$\beta_5(22) = 3$	$\beta_5(2111) = 6$	$\beta_5(3111) = 4$
$\beta_5(2) = 2$	$\beta_5(211) = 3$	$\beta_5(11111) = 12$	$\beta_5(222) = 4$
$\beta_5(11) = 2$	$\beta_5(1111) = 5$	$\beta_5(6) = 0$	$\beta_5(2211) = 6$
$\beta_5(3) = 0$	$\beta_5(5) = 0$	$\beta_5(51) = 0$	$\beta_5(21111) = 12$
$\beta_5(21) = 0$	$\beta_5(41) = 0$	$\beta_5(42) = 1$	$\beta_5(111111) = 24$
$\beta_5(111) = 2$	$\beta_5(32) = 0$	$\beta_5(411) = 1$	
$\beta_5(4) = 1$	$\beta_5(311) = 2$	$\beta_5(33) = 1$	

2.1.6. *Degree 6.*

$\beta_6() = 3$	$\beta_6(5) = 0$	$\beta_6(321) = 6$	$\beta_6(4111) = 5$
$\beta_6(1) = 0$	$\beta_6(41) = 0$	$\beta_6(3111) = 10$	$\beta_6(331) = 3$
$\beta_6(2) = 2$	$\beta_6(32) = 0$	$\beta_6(222) = 11$	$\beta_6(322) = 4$
$\beta_6(11) = 2$	$\beta_6(311) = 3$	$\beta_6(2211) = 16$	$\beta_6(3211) = 10$
$\beta_6(3) = 0$	$\beta_6(221) = 3$	$\beta_6(21111) = 30$	$\beta_6(31111) = 20$
$\beta_6(21) = 0$	$\beta_6(2111) = 10$	$\beta_6(111111) = 60$	$\beta_6(2221) = 14$
$\beta_6(111) = 2$	$\beta_6(11111) = 22$	$\beta_6(7) = 0$	$\beta_6(22111) = 30$
$\beta_6(4) = 2$	$\beta_6(6) = 1$	$\beta_6(61) = 0$	$\beta_6(211111) = 60$
$\beta_6(31) = 2$	$\beta_6(51) = 1$	$\beta_6(52) = 0$	$\beta_6(1111111) = 120$
$\beta_6(22) = 5$	$\beta_6(42) = 3$	$\beta_6(511) = 1$	
$\beta_6(211) = 5$	$\beta_6(411) = 3$	$\beta_6(43) = 0$	
$\beta_6(1111) = 8$	$\beta_6(33) = 3$	$\beta_6(421) = 2$	

2.1.7. *Degree 7.*

$\beta_7() = 4$	$\beta_7(2111) = 16$	$\beta_7(43) = 1$	$\beta_7(5111) = 6$
$\beta_7(1) = 0$	$\beta_7(11111) = 34$	$\beta_7(421) = 6$	$\beta_7(44) = 2$
$\beta_7(2) = 3$	$\beta_7(6) = 2$	$\beta_7(4111) = 15$	$\beta_7(431) = 5$
$\beta_7(11) = 3$	$\beta_7(51) = 2$	$\beta_7(331) = 10$	$\beta_7(422) = 9$
$\beta_7(3) = 0$	$\beta_7(42) = 6$	$\beta_7(322) = 12$	$\beta_7(4211) = 15$
$\beta_7(21) = 0$	$\beta_7(411) = 6$	$\beta_7(3211) = 30$	$\beta_7(41111) = 30$
$\beta_7(111) = 3$	$\beta_7(33) = 6$	$\beta_7(31111) = 60$	$\beta_7(332) = 10$
$\beta_7(4) = 3$	$\beta_7(321) = 12$	$\beta_7(2221) = 42$	$\beta_7(3311) = 20$
$\beta_7(31) = 3$	$\beta_7(3111) = 20$	$\beta_7(22111) = 90$	$\beta_7(3221) = 30$
$\beta_7(22) = 7$	$\beta_7(222) = 22$	$\beta_7(211111) = 180$	$\beta_7(32111) = 60$
$\beta_7(211) = 7$	$\beta_7(2211) = 32$	$\beta_7(1111111) = 360$	$\beta_7(311111) = 120$
$\beta_7(1111) = 11$	$\beta_7(21111) = 60$	$\beta_7(8) = 0$	$\beta_7(2222) = 48$
$\beta_7(5) = 0$	$\beta_7(111111) = 120$	$\beta_7(71) = 0$	$\beta_7(22211) = 90$
$\beta_7(41) = 0$	$\beta_7(7) = 0$	$\beta_7(62) = 1$	$\beta_7(221111) = 180$
$\beta_7(32) = 0$	$\beta_7(61) = 0$	$\beta_7(611) = 1$	$\beta_7(2111111) = 360$
$\beta_7(311) = 5$	$\beta_7(52) = 0$	$\beta_7(53) = 1$	$\beta_7(11111111) = 720$
$\beta_7(221) = 5$	$\beta_7(511) = 3$	$\beta_7(521) = 3$	

2.1.8. *Degree 8.* The symbol ? below means that on May 5, 1996 I did not know the relevant dimension.

		$\beta_8(221111) = 630$
$\beta_8() = 5$	$\beta_8(52) = 0$	$\beta_8(2111111) = ?$
$\beta_8(1) = 0$	$\beta_8(511) = 6$	$\beta_8(11111111) = ?$
$\beta_8(2) = 4$	$\beta_8(43) = 1$	$\beta_8(9) = 0$
$\beta_8(11) = 4$	$\beta_8(421) = 12$	$\beta_8(81) = 0$
$\beta_8(3) = 0$	$\beta_8(4111) = 32$	$\beta_8(72) = 0$
$\beta_8(21) = ?$	$\beta_8(331) = 19$	$\beta_8(711) = 1$
$\beta_8(111) = ?$	$\beta_8(322) = 24$	$\beta_8(63) = 1$
$\beta_8(4) = 4$	$\beta_8(3211) = 63$	$\beta_8(621) = 3$
$\beta_8(31) = ?$	$\beta_8(31111) = 131$	$\beta_8(6111) = 7$
$\beta_8(22) = ?$	$\beta_8(2221) = 88$	$\beta_8(54) = 1$
$\beta_8(211) = ?$	$\beta_8(22111) = 193$	$\beta_8(531) = 7$
$\beta_8(1111) = ?$	$\beta_8(211111) = ?$	$\beta_8(522) = 9$
$\beta_8(5) = 0$	$\beta_8(1111111) = ?$	$\beta_8(5211) = 21$
$\beta_8(41) = 0$	$\beta_8(8) = 1$	$\beta_8(51111) = 42$
$\beta_8(32) = 0$	$\beta_8(71) = 1$	$\beta_8(441) = 8$
$\beta_8(311) = 7$	$\beta_8(62) = 4$	$\beta_8(432) = 16$
$\beta_8(221) = ?$	$\beta_8(611) = 4$	$\beta_8(4311) = 35$
$\beta_8(2111) = ?$	$\beta_8(53) = 5$	$\beta_8(4221) = 51$
$\beta_8(11111) = ?$	$\beta_8(521) = 12$	$\beta_8(42111) = 105$
$\beta_8(6) = 3$	$\beta_8(5111) = 21$	$\beta_8(411111) = 210$
$\beta_8(51) = 4$	$\beta_8(44) = 8$	$\beta_8(333) = 24$
$\beta_8(42) = 11$	$\beta_8(431) = 19$	$\beta_8(3321) = 70$
$\beta_8(411) = 12$	$\beta_8(422) = 33$	$\beta_8(33111) = 140$
$\beta_8(33) = 11$	$\beta_8(4211) = 54$	$\beta_8(3222) = 102$
$\beta_8(321) = 22$	$\beta_8(41111) = 105$	$\beta_8(32211) = 210$
$\beta_8(3111) = 36$	$\beta_8(332) = 38$	$\beta_8(321111) = 420$
$\beta_8(222) = 39$	$\beta_8(3311) = 70$	$\beta_8(3111111) = 840$
$\beta_8(2211) = ?$	$\beta_8(3221) = 108$	$\beta_8(22221) = 312$
$\beta_8(21111) = ?$	$\beta_8(32111) = 210$	$\beta_8(222111) = 630$
$\beta_8(111111) = ?$	$\beta_8(311111) = 420$	$\beta_8(2211111) = 1260$
$\beta_8(7) = 0$	$\beta_8(2222) = 171$	$\beta_8(21111111) = 2520$
$\beta_8(61) = 0$	$\beta_8(22211) = 318$	$\beta_8(111111111) = 5040$

2.1.9. *Degree 9.* For degrees 9 and up, only the dimensions known as of May 5, 1996 are displayed, and two digit numbers in the partition u are underlined.

$\beta_9(1) = 0$	$\beta_9(44) = 20$	$\beta_9(91) = 0$	$\beta_9(5221) = 84$
$\beta_9(2) = 5$	$\beta_9(9) = 0$	$\beta_9(82) = 1$	$\beta_9(52111) = 168$
$\beta_9(11) = 5$	$\beta_9(81) = 0$	$\beta_9(811) = 1$	$\beta_9(511111) = 336$
$\beta_9(3) = 0$	$\beta_9(72) = 0$	$\beta_9(73) = 1$	$\beta_9(442) = 38$
$\beta_9(4) = 6$	$\beta_9(711) = 4$	$\beta_9(721) = 4$	$\beta_9(4411) = 70$
$\beta_9(5) = 0$	$\beta_9(63) = 3$	$\beta_9(7111) = 8$	$\beta_9(433) = 46$
$\beta_9(6) = 5$	$\beta_9(621) = 12$	$\beta_9(64) = 3$	$\beta_9(4321) = 140$
$\beta_9(7) = 0$	$\beta_9(6111) = 28$	$\beta_9(631) = 9$	$\beta_9(43111) = 280$
$\beta_9(61) = 0$	$\beta_9(54) = 4$	$\beta_9(622) = 16$	$\beta_9(4222) = 216$
$\beta_9(8) = 2$	$\beta_9(531) = 28$	$\beta_9(6211) = 28$	$\beta_9(42211) = 420$
$\beta_9(71) = 3$	$\beta_9(522) = 36$	$\beta_9(61111) = 56$	$\beta_9(3331) = 186$
$\beta_9(62) = 10$	$\beta_9(5211) = 84$	$\beta_9(55) = 3$	$\beta_9(3322) = 280$
$\beta_9(611) = 11$	$\beta_9(441) = 32$	$\beta_9(541) = 14$	
$\beta_9(53) = 13$	$\beta_9(432) = 64$	$\beta_9(532) = 28$	
$\beta_9(521) = 30$	$\beta_9(\underline{10}) = 0$	$\beta_9(5311) = 56$	

2.1.10. *Degree 10.*

$\beta_{10}(1) = 0$	$\beta_{10}(73) = 8$	$\beta_{10}(911) = 1$	$\beta_{10}(632) = 40$
$\beta_{10}(7) = 0$	$\beta_{10}(721) = 20$	$\beta_{10}(83) = 1$	$\beta_{10}(6311) = 84$
$\beta_{10}(8) = 4$	$\beta_{10}(7111) = 36$	$\beta_{10}(821) = 4$	$\beta_{10}(6221) = 124$
$\beta_{10}(71) = 6$	$\beta_{10}(64) = 16$	$\beta_{10}(8111) = 9$	$\beta_{10}(62111) = 252$
$\beta_{10}(81) = 0$	$\beta_{10}(631) = 44$	$\beta_{10}(74) = 2$	$\beta_{10}(611111) = 504$
$\beta_{10}(72) = 0$	$\beta_{10}(622) = 74$	$\beta_{10}(731) = 12$	$\beta_{10}(551) = 25$
$\beta_{10}(711) = 10$	$\beta_{10}(55) = 16$	$\beta_{10}(722) = 16$	$\beta_{10}(542) = 60$
$\beta_{10}(\underline{10}) = 1$	$\beta_{10}(541) = 66$	$\beta_{10}(7211) = 36$	$\beta_{10}(5411) = 126$
$\beta_{10}(91) = 1$	$\beta_{10}(\underline{11}) = 0$	$\beta_{10}(71111) = 72$	$\beta_{10}(533) = 84$
$\beta_{10}(82) = 5$	$\beta_{10}(\underline{10}1) = 0$	$\beta_{10}(65) = 3$	$\beta_{10}(5321) = 252$
$\beta_{10}(811) = 5$	$\beta_{10}(92) = 0$	$\beta_{10}(641) = 20$	$\beta_{10}(443) = 102$

2.1.11. *Degree 11.*

$\beta_{11}(8) = 8$	$\beta_{11}(83) = 5$	$\beta_{11}(921) = 5$	$\beta_{11}(7221) = 180$
$\beta_{11}(9) = 0$	$\beta_{11}(821) = 20$	$\beta_{11}(9111) = 10$	$\beta_{11}(72111) = 360$
$\beta_{11}(81) = 0$	$\beta_{11}(8111) = 45$	$\beta_{11}(84) = 5$	$\beta_{11}(66) = 9$
$\beta_{11}(\underline{10}) = 2$	$\beta_{11}(74) = 10$	$\beta_{11}(831) = 15$	$\beta_{11}(651) = 42$
$\beta_{11}(91) = 3$	$\beta_{11}(731) = 60$	$\beta_{11}(822) = 25$	$\beta_{11}(642) = 110$
$\beta_{11}(82) = 15$	$\beta_{11}(65) = 16$	$\beta_{11}(8211) = 45$	$\beta_{11}(6411) = 210$
$\beta_{11}(811) = 16$	$\beta_{11}(\underline{12}) = 0$	$\beta_{11}(81111) = 90$	$\beta_{11}(633) = 141$
$\beta_{11}(\underline{11}) = 0$	$\beta_{11}(\underline{11}1) = 0$	$\beta_{11}(75) = 6$	$\beta_{11}(552) = 126$
$\beta_{11}(\underline{10}1) = 0$	$\beta_{11}(\underline{10}2) = 1$	$\beta_{11}(741) = 30$	
$\beta_{11}(92) = 0$	$\beta_{11}(\underline{10}11) = 1$	$\beta_{11}(732) = 60$	
$\beta_{11}(911) = 5$	$\beta_{11}(93) = 2$	$\beta_{11}(7311) = 120$	

2.1.12. *Degree 12.*

$\beta_{12}(\underline{10}) = 5$	$\beta_{12}(921) = 30$	$\beta_{12}(\underline{10}3) = 1$	$\beta_{12}(85) = 7$
$\beta_{12}(\underline{10}1) = 0$	$\beta_{12}(9111) = 55$	$\beta_{12}(\underline{10}21) = 5$	$\beta_{12}(841) = 40$
$\beta_{12}(92) = 0$	$\beta_{12}(84) = 29$	$\beta_{12}(\underline{10}111) = 11$	$\beta_{12}(832) = 80$
$\beta_{12}(\underline{12}) = 1$	$\beta_{12}(75) = 38$	$\beta_{12}(94) = 3$	$\beta_{12}(8311) = 165$
$\beta_{12}(\underline{11}1) = 1$	$\beta_{12}(\underline{13}) = 0$	$\beta_{12}(931) = 18$	$\beta_{12}(76) = 9$
$\beta_{12}(\underline{10}2) = 6$	$\beta_{12}(\underline{12}1) = 0$	$\beta_{12}(922) = 25$	$\beta_{12}(751) = 66$
$\beta_{12}(\underline{10}11) = 6$	$\beta_{12}(\underline{11}2) = 0$	$\beta_{12}(9211) = 55$	$\beta_{12}(661) = 75$
$\beta_{12}(93) = 12$	$\beta_{12}(\underline{11}11) = 1$	$\beta_{12}(91111) = 110$	

2.1.13. *Degree 13.*

$\beta_{13}(\underline{11}) = 0$	$\beta_{13}(\underline{10}3) = 8$	$\beta_{13}(\underline{11}3) = 2$	$\beta_{13}(\underline{10}1111) = 132$
$\beta_{13}(\underline{12}) = 3$	$\beta_{13}(\underline{10}21) = 30$	$\beta_{13}(\underline{11}21) = 6$	$\beta_{13}(95) = 11$
$\beta_{13}(\underline{11}1) = 4$	$\beta_{13}(94) = 20$	$\beta_{13}(\underline{11}111) = 12$	$\beta_{13}(941) = 55$
$\beta_{13}(\underline{13}) = 0$	$\beta_{13}(\underline{14}) = 0$	$\beta_{13}(\underline{10}4) = 7$	$\beta_{13}(932) = 110$
$\beta_{13}(\underline{12}1) = 0$	$\beta_{13}(\underline{13}1) = 0$	$\beta_{13}(\underline{10}31) = 22$	$\beta_{13}(86) = 19$
$\beta_{13}(\underline{11}2) = 0$	$\beta_{13}(\underline{12}2) = 1$	$\beta_{13}(\underline{10}22) = 36$	$\beta_{13}(851) = 99$
$\beta_{13}(\underline{11}11) = 6$	$\beta_{13}(\underline{12}11) = 1$	$\beta_{13}(\underline{10}211) = 66$	$\beta_{13}(77) = 19$

2.1.14. *Degree 14.*

$\beta_{14}(\underline{12}) = 7$	$\beta_{14}(\underline{11}3) = 16$	$\beta_{14}(\underline{12}3) = 2$	$\beta_{14}(\underline{11}211) = 78$
$\beta_{14}(\underline{12}1) = 0$	$\beta_{14}(\underline{11}21) = 42$	$\beta_{14}(\underline{12}21) = 6$	$\beta_{14}(\underline{11}1111) = 156$
$\beta_{14}(\underline{14}) = 1$	$\beta_{14}(\underline{15}) = 0$	$\beta_{14}(\underline{12}111) = 13$	$\beta_{14}(\underline{10}5) = 13$
$\beta_{14}(\underline{13}1) = 1$	$\beta_{14}(\underline{14}1) = 0$	$\beta_{14}(\underline{11}4) = 5$	$\beta_{14}(\underline{10}41) = 70$
$\beta_{14}(\underline{12}2) = 7$	$\beta_{14}(\underline{13}2) = 0$	$\beta_{14}(\underline{11}31) = 26$	$\beta_{14}(96) = 22$
$\beta_{14}(\underline{12}11) = 7$	$\beta_{14}(\underline{13}11) = 1$	$\beta_{14}(\underline{11}22) = 36$	$\beta_{14}(87) = 28$

2.1.15. *Degree 15.*

$\beta_{15}(\underline{13}) = 0$	$\beta_{15}(\underline{12}3) = 12$	$\beta_{15}(\underline{13}21) = 7$	$\beta_{15}(\underline{11}5) = 18$
$\beta_{15}(\underline{14}) = 3$	$\beta_{15}(\underline{12}21) = 42$	$\beta_{15}(\underline{13}111) = 14$	$\beta_{15}(\underline{11}41) = 91$
$\beta_{15}(\underline{13}1) = 5$	$\beta_{15}(\underline{16}) = 0$	$\beta_{15}(\underline{12}4) = 9$	$\beta_{15}(\underline{10}6) = 36$
$\beta_{15}(\underline{15}) = 0$	$\beta_{15}(\underline{15}1) = 0$	$\beta_{15}(\underline{12}31) = 30$	$\beta_{15}(97) = 47$
$\beta_{15}(\underline{14}1) = 0$	$\beta_{15}(\underline{14}2) = 1$	$\beta_{15}(\underline{12}22) = 49$	$\beta_{15}(88) = 58$
$\beta_{15}(\underline{13}2) = 0$	$\beta_{15}(\underline{14}11) = 1$	$\beta_{15}(\underline{12}211) = 91$	
$\beta_{15}(\underline{13}11) = 7$	$\beta_{15}(\underline{13}3) = 2$	$\beta_{15}(\underline{12}1111) = 182$	

2.1.16. *Degree 16.*

$\beta_{16}(\underline{14}) = 8$	$\beta_{16}(\underline{13}3) = 21$	$\beta_{16}(\underline{14}21) = 7$	$\beta_{16}(\underline{13}1111) = 210$
$\beta_{16}(\underline{14}1) = 0$	$\beta_{16}(\underline{17}) = 0$	$\beta_{16}(\underline{14}111) = 15$	$\beta_{16}(\underline{12}5) = 21$
$\beta_{16}(\underline{16}) = 1$	$\beta_{16}(\underline{16}1) = 0$	$\beta_{16}(\underline{13}4) = 7$	$\beta_{16}(\underline{11}6) = 42$
$\beta_{16}(\underline{15}1) = 1$	$\beta_{16}(\underline{15}2) = 0$	$\beta_{16}(\underline{13}31) = 35$	
$\beta_{16}(\underline{14}2) = 8$	$\beta_{16}(\underline{15}11) = 1$	$\beta_{16}(\underline{13}22) = 49$	
$\beta_{16}(\underline{14}11) = 8$	$\beta_{16}(\underline{14}3) = 2$	$\beta_{16}(\underline{13}211) = 105$	

2.1.17. *Degree 17.*

$\beta_{17}(\underline{15}) = 0$	$\beta_{17}(\underline{15}11) = 8$	$\beta_{17}(\underline{15}3) = 3$	$\beta_{17}(\underline{14}211) = 120$
$\beta_{17}(\underline{16}) = 3$	$\beta_{17}(\underline{14}3) = 16$	$\beta_{17}(\underline{15}21) = 8$	$\beta_{17}(\underline{14}1111) = 240$
$\beta_{17}(\underline{15}1) = 5$	$\beta_{17}(\underline{18}) = 0$	$\beta_{17}(\underline{15}111) = 16$	$\beta_{17}(\underline{13}5) = 28$
$\beta_{17}(\underline{17}) = 0$	$\beta_{17}(\underline{17}1) = 0$	$\beta_{17}(\underline{14}4) = 12$	
$\beta_{17}(\underline{16}1) = 0$	$\beta_{17}(\underline{16}2) = 1$	$\beta_{17}(\underline{14}31) = 40$	
$\beta_{17}(\underline{15}2) = 0$	$\beta_{17}(\underline{16}11) = 1$	$\beta_{17}(\underline{14}22) = 64$	

2.1.18. *Degree 18.*

$\beta_{18}(\underline{16}) = 10$	$\beta_{18}(\underline{16}11) = 9$	$\beta_{18}(\underline{17}11) = 1$	$\beta_{18}(\underline{15}31) = 45$
$\beta_{18}(\underline{16}1) = 0$	$\beta_{18}(\underline{15}3) = 27$	$\beta_{18}(\underline{16}3) = 2$	$\beta_{18}(\underline{15}22) = 64$
$\beta_{18}(\underline{18}) = 1$	$\beta_{18}(\underline{19}) = 0$	$\beta_{18}(\underline{16}21) = 8$	$\beta_{18}(\underline{15}211) = 136$
$\beta_{18}(\underline{17}1) = 1$	$\beta_{18}(\underline{18}1) = 0$	$\beta_{18}(\underline{16}111) = 17$	$\beta_{18}(\underline{15}1111) = 272$
$\beta_{18}(\underline{16}2) = 9$	$\beta_{18}(\underline{17}2) = 0$	$\beta_{18}(\underline{15}4) = 9$	$\beta_{18}(\underline{14}5) = 32$

2.1.19. *Degree 19.*

$\beta_{19}(\underline{17}) = 0$	$\beta_{19}(\underline{17}2) = 0$	$\beta_{19}(\underline{18}11) = 1$	$\beta_{19}(\underline{16}31) = 51$
$\beta_{19}(\underline{18}) = 4$	$\beta_{19}(\underline{17}11) = 9$	$\beta_{19}(\underline{17}3) = 3$	$\beta_{19}(\underline{16}22) = 81$
$\beta_{19}(\underline{17}1) = 6$	$\beta_{19}(\underline{20}) = 0$	$\beta_{19}(\underline{17}21) = 9$	$\beta_{19}(\underline{16}211) = 153$
$\beta_{19}(\underline{19}) = 0$	$\beta_{19}(\underline{19}1) = 0$	$\beta_{19}(\underline{17}111) = 18$	$\beta_{19}(\underline{15}5) = 41$
$\beta_{19}(\underline{18}1) = 0$	$\beta_{19}(\underline{18}2) = 1$	$\beta_{19}(\underline{16}4) = 15$	

2.1.20. *Degree 20.*

$\beta_{20}(\underline{18}1) = 0$	$\beta_{20}(\underline{21}) = 0$	$\beta_{20}(\underline{18}21) = 9$	$\beta_{20}(\underline{17}211) = 171$
$\beta_{20}(\underline{20}) = 1$	$\beta_{20}(\underline{20}1) = 0$	$\beta_{20}(\underline{18}111) = 19$	$\beta_{20}(\underline{16}5) = 46$
$\beta_{20}(\underline{19}1) = 1$	$\beta_{20}(\underline{19}2) = 0$	$\beta_{20}(\underline{17}4) = 12$	
$\beta_{20}(\underline{18}2) = 10$	$\beta_{20}(\underline{19}11) = 1$	$\beta_{20}(\underline{17}31) = 57$	
$\beta_{20}(\underline{18}11) = 10$	$\beta_{20}(\underline{18}3) = 3$	$\beta_{20}(\underline{17}22) = 81$	

2.1.21. *Remarks.*

Remark 4. The $\beta_m(u)$'s were computed by a rather complicated mathematica [11] program. Mathematica is a very expressive language, but it is rather slow. So some of the harder parts of the computation of some of the harder to compute $\beta_m(u)$'s were farmed out (through MathLink [12]) to programs written in C and C++. In particular, the problem of deciding whether two graphs are the same or not was farmed out to B. D. McKay's graph isomorphism and automorphism program `nauty` [8] through a custom-made MathLink interface. I wish to thank Prof. McKay for his assistance.

Remark 5. The program used to compute $\beta_m(u)$ is not yet available on my ftp site. Sorry, it is way too messy at this stage. The numbers $\beta_m(u)$ them self are available, in the file `table.m.gz` in my ftp site. See section 4.

Remark 6. There is one known bug in the that program, but it is unlikely that it caused a mistake in the end result. Roughly speaking, the problem is as follows. Let G be the graph whose vertices are all Chinese characters (corresponding to some fixed m and u), and whose edges correspond to $I \to H$ and $I \to X$ moves. Let G' be the graph obtained from G by removing all Chinese characters that have an automorphism which reverses the orientation of an odd number of trivalent vertices (that is, the Chinese characters that vanish already mod AS alone), and all edges connected to them. If G' is not connected, my program *may* miss a part of the space and give a too low dimension. The problem is technical and easy to fix, and, in fact, I also have a corrected version of the program in which the problem doesn't exist. But for some technical reason, the fixed version is much slower, and so, except for a small number of test cases (on which the two versions of the program happily agreed), most results were obtained with the wrong program. Generally speaking, G is highly connected and G' is obtained from it by removing a small number of 'randomly placed' vertices, and so I believe the results reported here are correct.

Remark 7. In degree 7 and up, some of the computations were performed modulo a large prime rather than over the rationals, to prevent overflow problems. It is unlikely that this affected the results in any way.

Remark 8. Just for the record, it took about 8 months of CPU time on a 1992 workstation to compute the above 614 dimensions, and they add up to a total of 30743.

Remark 9. In [2] it is proven that $\beta_m(1^{m+1}) = (m-1)!$.

Remark 10. It is easy to check that $\beta_m(m) = 1$ for even m, and $\beta_m(m) = 0$ for odd m.

Remark 11. With some effort one can prove that $\beta_m(m-1) = \lceil \frac{m}{6} \rceil$ for odd m, and $\beta_m(m-1) = 0$ for even m, where $\lceil x \rceil$ denotes the smallest integer not smaller than x.

2.2. The representation $\mathcal{E}_n$. The symmetric group S_n acts in a natural way on the space of Chinese characters having exactly n univalent vertices colored by n different colors (modulo IHX and AS). Knowing the dimensions $\beta_m(u)$ for all m and u is equivalent to knowing the decomposition of the resulting graded representation $\mathcal{E}_n$ of S_n into irreducibles, and the $\beta_m(u)$'s shown above are sufficient to determine the structure of $\mathcal{E}_n$ for some small values of n and in some small degrees. With R_u denoting the irreducible representation of S_n corresponding to the partition u of n (so that R_n and R_{1^n} are the trivial and the alternating representations respectively), here's what we can do:

$$
\begin{array}{lcl}
\mathcal{G}_1\mathcal{E}_2 &=& R_2 \\
\mathcal{G}_2\mathcal{E}_2 &=& R_2 \\
\mathcal{G}_3\mathcal{E}_2 &=& R_2 \\
\mathcal{G}_4\mathcal{E}_2 &=& R_2 \\
\mathcal{G}_5\mathcal{E}_2 &=& 2\,R_2 \\
\mathcal{G}_6\mathcal{E}_2 &=& 2\,R_2 \\
\mathcal{G}_7\mathcal{E}_2 &=& 3\,R_2 \\
\mathcal{G}_8\mathcal{E}_2 &=& 4\,R_2 \\
\mathcal{G}_9\mathcal{E}_2 &=& 5\,R_2 \\
\mathcal{G}_2\mathcal{E}_3 &=& R_{111} \\
\mathcal{G}_3\mathcal{E}_3 &=& R_{111} \\
\mathcal{G}_4\mathcal{E}_3 &=& R_{111} \\
\mathcal{G}_5\mathcal{E}_3 &=& 2\,R_{111} \\
\mathcal{G}_6\mathcal{E}_3 &=& 2\,R_{111} \\
\mathcal{G}_7\mathcal{E}_3 &=& 3\,R_{111} \\
\mathcal{G}_3\mathcal{E}_4 &=& R_{22} \\
\mathcal{G}_4\mathcal{E}_4 &=& R_4 + R_{22} \\
\mathcal{G}_5\mathcal{E}_4 &=& R_4 + 2\,R_{22}
\end{array}
\qquad
\begin{array}{lcl}
\mathcal{G}_6\mathcal{E}_4 &=& 2\,R_4 + 3\,R_{22} \\
\mathcal{G}_7\mathcal{E}_4 &=& 3\,R_4 + 4\,R_{22} \\
\mathcal{G}_4\mathcal{E}_5 &=& R_{311} \\
\mathcal{G}_5\mathcal{E}_5 &=& 2\,R_{311} \\
\mathcal{G}_6\mathcal{E}_5 &=& 3\,R_{311} + R_{2111} \\
\mathcal{G}_7\mathcal{E}_5 &=& 5\,R_{311} + R_{2111} \\
\mathcal{G}_5\mathcal{E}_6 &=& R_{42} + R_{222} + R_{3111} \\
\mathcal{G}_6\mathcal{E}_6 &=& R_6 + 2\,R_{42} + 2\,R_{222} + R_{321} + R_{3111} + R_{21111} \\
\mathcal{G}_7\mathcal{E}_6 &=& 2\,R_6 + 4\,R_{42} + 4\,R_{222} + 2\,R_{321} + 2\,R_{3111} + 2\,R_{21111} \\
\mathcal{G}_6\mathcal{E}_7 &=& R_{331} + R_{421} + R_{511} + R_{3211} + R_{22111} \\
\mathcal{G}_7\mathcal{E}_7 &=& R_{43} + 3\,R_{331} + 2\,R_{421} + 3\,R_{511} + R_{2221} + 3\,R_{3211} \\
&& + R_{4111} + 2\,R_{22111} + R_{1111111} \\
\mathcal{G}_7\mathcal{E}_8 &=& R_{44} + R_{62} + 2\,R_{422} + R_{431} + R_{521} + R_{2222} + R_{3221} \\
&& + 2\,R_{3311} + R_{4211} + R_{5111} + R_{32111} + R_{41111} + R_{221111} \\
\mathcal{G}_8\mathcal{E}_9 &=& R_{63} + 2\,R_{333} + 2\,R_{432} + R_{441} + R_{522} + 3\,R_{531} \\
&& + R_{621} + R_{711} + R_{3222} + 2\,R_{3321} + 3\,R_{4221} + 3\,R_{4311} \\
&& + 3\,R_{5211} + R_{6111} + 3\,R_{32211} + R_{33111} + 3\,R_{42111} \\
&& + R_{222111} + R_{321111} + R_{411111} + R_{3111111}
\end{array}
$$

Remark 12. As we are restricting our attention to connected Chinese Characters, it is clear that $\mathcal{G}_n\mathcal{E}_m = 0$ if $n < m - 1$.

Remark 13. Notice that many of the possible irreducible representations of S_1–S_9 are missing in the above decompositions. I do not know why this is so.

Remark 14. Kontsevich's observation (see remark 2) implies that $\mathcal{E}_1 = 0$ and that $\mathcal{E}_2$ is a multiple of R_2. Vogel [10] proved that R_{111} appears in $\mathcal{E}_3$ with the same graded multiplicity as R_2 in $\mathcal{E}_2$ (but it is not known whether $\mathcal{E}_3$ is a multiple of R_{111}). The degree m multiplicity in both cases is $\beta_{m,2} = \beta_{m-1,0}$.

Remark 15. The decompositions presented here were computed from the $\beta_m(u)$'s using Stembridge's maple package `SF` [9]. I wish to thank N. Bergeron for his help with both the mathematical and the computational aspects of this computation.

2.3. Invariants.

2.3.1. *Primitive invariants.* The information in the lists 2.1.1–2.1.7 can be used to determine the dimension of the space of type m primitive Vassiliev invariants of n-component string links, for sufficiently small values of m:

$m\backslash n$	1	2	3	4	5	6	general n
1	1	3	6	10	15	21	$\frac{n(1+n)}{2}$
2	1	3	7	14	25	41	$\frac{n\left(5+n^2\right)}{6}$
3	1	4	13	34	75	146	$\frac{n\left(10-n+2n^2+n^3\right)}{12}$
4	2	9	34	105	271	608	$\frac{n\left(154+45n+20n^2+15n^3+6n^4\right)}{120}$
5	3	16	81	321	1012	2679	$\frac{n\left(258+31n+30n^2+25n^3+12n^4+4n^5\right)}{120}$
6	5	32	208	1040	3987	12452	$\frac{n\left(1208+490n+175n^2+105n^3+77n^4+35n^5+10n^6\right)}{420}$
7	8	62	547	3515	16469	60767	$\frac{n\left(4042+1175n+700n^2+350n^3+238n^4+140n^5+60n^6+15n^7\right)}{840}$

Remark 16. The dimensions presented above (and in sections 2.3.2, 2.3.3 and 3) are of the appropriate degree subspace of the associated graded space. Remember that a Vassiliev invariant of type 2 is also of type 4, and thus the total dimension of the space of type 4 (say) primitive Vassiliev invariants of 3-component (say) string links is $1 + 6 + 7 + 13 + 34 = 61$, where the first summand (1) comes from the type 0 (constant) invariants, not listed in the above table.

2.3.2. *Framed invariants.* The information about primitive invariants can be assembled to give the dimension of the space of all Vassiliev invariants of type m of n-component framed string links (excluding invariants of lower type):

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$m=1$	1	3	6	10	15	21
$m=2$	2	9	28	69	145	272
$m=3$	3	23	111	394	1130	2778
$m=4$	6	60	413	2035	7781	24632
$m=5$	10	148	1461	9849	49455	198981
$m=6$	19	366	5027	45680	297622	1506218
$m=7$	33	884	16924	205612	1722724	10875542

The formulas for general n are (the formula for $m=7$ is too messy to print):

m	type m invariants (excluding lower types)
1	$\frac{n\,(1+n)}{2}$
2	$\frac{n\left(26+9\,n+10\,n^2+3\,n^3\right)}{24}$
3	$\frac{n\left(48+30\,n+41\,n^2+17\,n^3+7\,n^4+n^5\right)}{48}$
4	$\frac{n\left(10512+9140\,n+6900\,n^2+4415\,n^3+2568\,n^4+830\,n^5+180\,n^6+15\,n^7\right)}{5760}$
5	$\frac{n\,(2+n)\,(3+n)\left(4320+920\,n+2682\,n^2+805\,n^3+638\,n^4+192\,n^5+40\,n^6+3\,n^7\right)}{11520}$
6	$\frac{n\left(10607616+13964832\,n+11387656\,n^2+8527428\,n^3+5541270\,n^4+3139353\,n^5+1376706\,n^6+477729\,n^7+115514\,n^8+17955\,n^9+1638\,n^{10}+63\,n^{11}\right)}{2903040}$

2.3.3. *Unframed invariants.* Similarly, here are some dimensions of spaces of Vassiliev invariants of type m of n-component unframed string links (excluding invariants of lower type):

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$m=1$	0	1	3	6	10	15
$m=2$	1	4	13	35	80	161
$m=3$	1	8	44	174	545	1441
$m=4$	3	23	158	834	3436	11639
$m=5$	4	51	527	3807	20474	87728
$m=6$	9	130	1772	16987	117567	630207
$m=7$	14	300	5813	74240	656859	4370368

The formulas for general n are (the formula for $m = 7$ is too messy to print):

m	type m invariants (excluding lower types)
1	$\frac{(-1+n)\,n}{2}$
2	$\frac{n\,(1+n)\left(14-5\,n+3\,n^2\right)}{24}$
3	$\frac{n\left(32-10\,n+15\,n^2+9\,n^3+n^4+n^5\right)}{48}$
4	$\frac{n\left(9072+1940\,n+4380\,n^2+575\,n^3+888\,n^4+350\,n^5+60\,n^6+15\,n^7\right)}{5760}$
5	$\frac{n\left(23616-464\,n+11532\,n^2+4680\,n^3+4489\,n^4+1371\,n^5+658\,n^6+170\,n^7+25\,n^8+3\,n^9\right)}{11520}$
6	$\frac{n\left(10123776+3844512\,n+6322456\,n^2+2320164\,n^3+1893570\,n^4+931833\,n^5+485382\,n^6+153153\,n^7+43694\,n^8+7875\,n^9+882\,n^{10}+63\,n^{11}\right)}{2903040}$

2.3.4. *Factoring knots out.* Divide $\mathcal{B}^{sl}$ further by the ideal generated by 'boring' connected Chinese character — Chinese characters all of whose univalent vertices are colored by the same color. The resulting dimensions ("invariants of string links that do not have a factor that depends on only a single component", or, "genuine string link invariants") are:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$m=1$	0	1	3	6	10	15
$m=2$	0	2	10	31	75	155
$m=3$	0	4	32	146	490	1345
$m=4$	0	10	107	668	2986	10586
$m=5$	0	22	347	2963	17359	78179
$m=6$	0	53	1132	12923	97787	552517
$m=7$	0	121	3653	55576	538394	3781984

3. Braids

Just for the sake of comparison, here are some dimensions of spaces of Vassiliev invariants of braids. Notice that even the dimensions in section 2.3.4 are bigger than the dimensions below.

$m\backslash n$	1	2	3	4	5	6	general n
1	0	1	3	6	10	15	$\frac{(-1+n)\,n}{2}$
2	0	1	7	25	65	140	$\frac{(-1+n)\,n\,(1+n)\,(-2+3\,n)}{24}$
3	0	1	15	90	350	1050	$\frac{(-1+n)^2\,n^2\,(1+n)\,(2+n)}{48}$
4	0	1	31	301	1701	6951	$\frac{(-1+n)\,n\,(1+n)\,(2+n)\,(3+n)\left(8-10\,n-15\,n^2+15\,n^3\right)}{5760}$
5	0	1	63	966	7770	42525	$\frac{(-1+n)^2\,n^2\,(1+n)\,(2+n)\,(3+n)\,(4+n)\left(-6+n+3\,n^2\right)}{11520}$
6	0	1	127	3025	34105	246730	$\frac{(-1+n)\,n\,(1+n)\,(2+n)\,(3+n)\,(4+n)\,(5+n)\left(-96+140\,n+224\,n^2-315\,n^3+63\,n^5\right)}{2903040}$
7	0	1	255	9330	145750	1379400	too messy to print

Remark 17. These dimensions $d(m,n)$ satisfy $d(m,n) = (n-1)d(m-1,n)+d(m,n-1)$ for $m \geq 1$ and $n \geq 2$, $d(0,n) = 1$ for $n \geq 1$ and $d(m,1) = 0$ for $m \geq 1$. This fact, which makes their computation very easy, follows from Drinfel'd's presentation of (his notation) $\mathfrak{a}_n$ as a semidirect product of free algebras [4, page 847]. See also [3, 5, 7].

4. The data file

The file `table.m.gz` in my ftp site contains, in a mathematica readable format, some of the data presented in this article as well as explicit basis for some of the spaces whose dimensions were given here. To read this file, get it from my ftp site, uncompress it by typing `gunzip table.m` at the UNIX prompt, start up mathematica and load it in:

```
Mathematica 2.2 for SPARC
Copyright 1988-93 Wolfram Research, Inc.
 -- Open Look graphics initialized --
```

In[1]:= `<< table.m`

The space $\mathcal{B}_5(22)$ is of dimension 3. Indeed,

In[2]:= `DimB[5,{2,2}]`

Out[2]= 3

A basis of $\mathcal{B}_3(22)$ (which is 1-dimensional) is

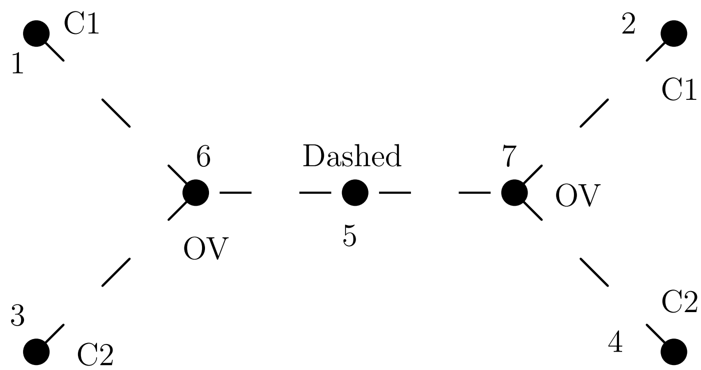
In[3]:= `BBasis[3,{2,2}] // InputForm`

Out[3]=

```
{Graph[C1[6], C1[7], C2[6], C2[7], Dashed[6,7], OV[1,3,5], OV[2,4,5]]}
```

Let us explain the notation: A `Graph` is an object of the form `Graph[`$v_1, v_2, \ldots$`]`, where each v_j is a vertex, that is, an object of the form $vname_j$`[`$p_{j1}, p_{j2}, \ldots$`]`. Such a `Graph` object represents a graph with numbered and named vertices, so that the jth vertex is named $vname_j$, and is connected by an arc to the vertices numbered $p_{j1}, p_{j2}, \ldots$. Chinese Characters are represented by such `Graph` object containing three types of vertices — oriented trivalent vertices named `OV`, colored univalent vertices named `C`i for some integer i, and artificial bivalent vertices named `Dashed` placed in the middle of each dashed line connecting two trivalent vertices.

Therefore, the Chinese Character in *Out[3]* is:



5. Acknowledgement

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Department of Mathematics, Harvard University, Cambridge, MA 02138

Current address: Institute of Mathematics, The Hebrew University, Giv'at-Ram, Jerusalem 91904, Israel

E-mail address: drorbn@math.huji.ac.il