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Weights of Feynman diagrams and the Vassiliev knot invariants

Dror Bar-Natan[1]

Abstract

Given a representation of a Lie algebra and an *ad*-invariant bilinear form we show how to assign numerical weights to a certain collection of graphs. This assignment is then shown to satisfy certain relations first written by Birman and Lin as consistency conditions for the Vassiliev procedure for obtaining knot invariants. Our considerations are motivated by the combinatorics underlying the perturbative expansion of the Chern-Simons quantum field theory. We discuss the set of all solutions of these relations, whose importance stems from the fact that each such solution should correspond via perturbation theory to a link invariant.

[1]Department of Mathematics, Harvard University, Cambridge MA 02138.
E-mail: dror@math.harvard.edu.

1 Introduction

The purpose of this paper is to introduce a certain combinatorial-algebraic problem, discuss its significance to knot theory (and to a lesser extent, to quantum field theory), and present some solutions to it. The most general solution to this problem has not yet been found, and finding it is likely to lead to the discovery of new knot and link invariants.

In this paper, the words *closed diagram* will always refer to a graph made of a certain number of directed ellipses (called *Wilson loops*) marked by the natural numbers $1, \ldots, I$, and a certain number of dashed lines (called *propagators*). The propagators and the Wilson loops are allowed to meet in two types of vertices — one type (called *type* $R^2\mathcal{G}$) in which a propagator ends on one of the Wilson loops, and another (called *type* $\mathcal{G}^3$) connecting three propagators. We assume that the second kind of vertices are 'oriented' — that one of the two possible cyclic orders of the three propagators meeting in such a vertex is specified. The 'order' of such a diagrams will be half the total number of vertices in it.

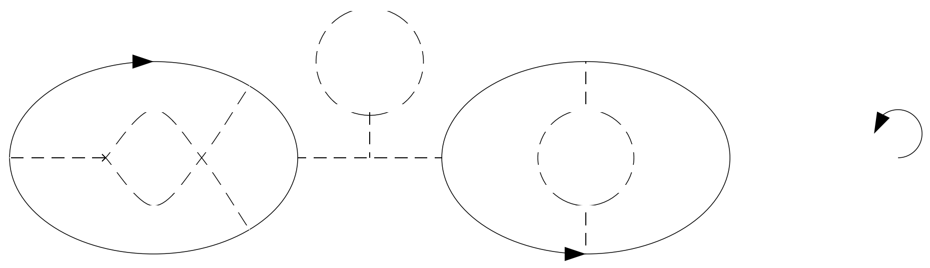


Figure 1. An example for a closed diagram of order 6.

Figure 1 is an example for such a diagram with $I = 2$. In this figure (as in the rest of this paper) each of the vertices is oriented *counterclockwise* ($\circlearrowleft$). This convention means that the two diagram parts in figure 2 are *not* equivalent. Also, remember that our diagrams are not allowed to have higher than cubic vertices. It is therefore implicitly understood that when four or more lines meet at the same point, that point is not a vertex and those lines pass each other without "interaction".

Figure 2. Two diagram parts which differ only in the orientation of one of their vertices.

We will be looking for assignments $D \to C(D)$ that assign a weight $C(D)$ inside some pre-chosen abelian group to each diagram D, and satisfy the following two relations:
The "IHX" relation: Let the diagrams I, H, and X be identical outside a small domain, inside which they look as in figure 3. Then their weights are expected to satisfy

$$C(I) = C(H) - C(X).$$

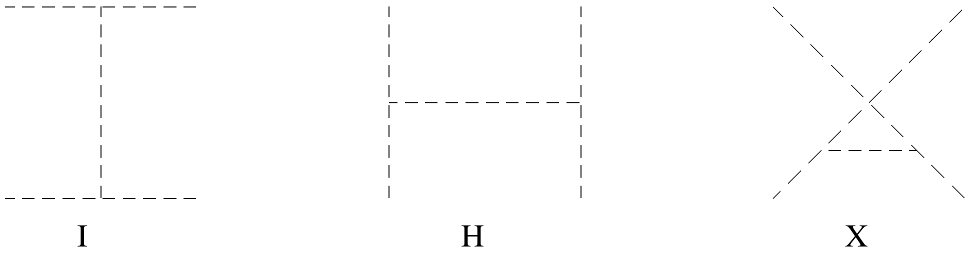


Figure 3. The diagrams I, H, and X.

The "STU" relation: Let the diagrams S, T, and U be identical outside a small domain, inside which they look as in figure 4. Then their weights are expected to satisfy

$$C(S) = C(T) - C(U).$$

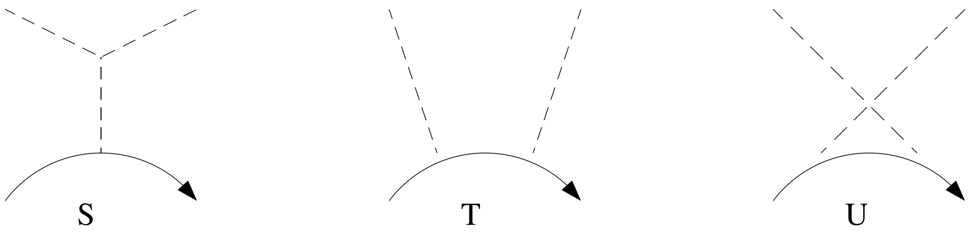


Figure 4. The diagrams S, T, and U.

Main problem Find all such assignments C.

Such assignments will be called *weight systems.*

There are very good reasons to believe that each weight system will give rise to a link invariant. When one considers the perturbative expansion of the Chern-Simons quantum field theory [2, 6], one encounters diagrams much like the above. The diagrams in the Chern-Simons theory correspond to integrals, and I have shown in [3] that (assuming some convergence which is yet to be proven) these integrals summed with 'correct' weights add up to give link invariants. The word 'correct' in the previous sentence means exactly "satisfying the relations IHX and STU". In [2] I have carried out this program for the diagrams of order ≤ 2, and in [10, 11] Witten has shown that the HOMFLY polynomial [7] can be derived from the Chern-Simons quantum field theory, and therefore can probably be re-derived using our techniques. The weight system C that should correspond to the HOMFLY polynomial is presented in section 6. I don't know which are the knot invariants corresponding to most of the other weight systems presented in this paper, and I do not know whether there are further weight systems beyond those presented here.

As was (implicitly) shown in [10] and discussed in [2, 3] from the perturbative point of view, to each weight system should correspond a three-manifold invariant as well.

In section 7 a second relation, due to Vassiliev [9] and Birman-Lin [4], between those weight systems and knot theory is discussed.

2 The method

Let $\mathbf{F}$ be a field, and let D be a closed diagram. I will now show how, given some Lie algebraic data, we can associate an element $C_{\mathcal{G}}(D)$ of $\mathbf{F}$ to D.

Let $\mathcal{G}$ be a finite dimensional Lie algebra over the field $\mathbf{F}$, $R_1, \ldots, R_I$ a list of finite dimensional representations of $\mathcal{G}$ (one for each Wilson loop in D) of dimensions $d_1, \ldots, d_I$, and let tr be a non-degenerate $\mathbf{F}$-valued ad-invariant bilinear form on $\mathcal{G}\otimes\mathcal{G}$, where ad denotes the adjoint representation of the Lie algebra $\mathcal{G}$ on its underlying vector space. Let $\{\mathcal{G}_a\}$ be a basis for $\mathcal{G}$, $\{r_i^\alpha\}$ a basis of R_i, and define the tensors t_{ab}, t^{bc}, f_{ab}^c, t_{abc}, and $R_{ia\beta}^\alpha$ by the following formulae:

$$\begin{aligned}
t_{ab} &= tr(\mathcal{G}_a, \mathcal{G}_b), \\
t_{ab}t^{bc} &= \delta_a^{\ c}, \\
[\mathcal{G}_a, \mathcal{G}_b] &= f_{ab}^c \mathcal{G}_c, \\
t_{abc} &= f_{ab}^d t_{dc}, \\
R_i(\mathcal{G}_a) r_i^\alpha &= R_{ia\beta}^\alpha r_i^\beta.
\end{aligned}$$

(In all those formulae the Einstein summation convention is assumed — there is an implicit summation over every index that is repeated twice in a formula, once as an upper index and once as a lower).

To define $C_{\mathcal{G}}(D)$, first mark every Wilson loop segment in D by a greek letter $\alpha, \beta, \ldots$, and every end of every propagator by a small letter in the English alphabet — a, b, ….

Figure 5. An unmarked diagram and a marked diagram.

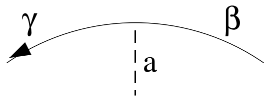
I will now describe how to construct a certain algebraic expression out of D and its marking:

1. To each type $\mathcal{G}^3$ vertex in D associate a $t_{\ldots}$ symbol with the $\cdots$ replaced by the letters marking that vertex, picking those letter in an order consistent with the orientation of the vertex. Using the invariance of t_{ab} it is easy to check that $t_{abc} = t_{bca} = t_{cab}$, and so the particular order chosen is immaterial.

2. To each propagator in D associate a $t^{\cdot\cdot}$ symbol with the dots replaced by the letters marked at the ends of that propagator.

3. To each type $R^2\mathcal{G}$ vertex associate an $R^{\cdot}_{\cdot\cdot}$ symbol with the dots replaced by the letters

marking that vertex, as in the figure below:



$\longrightarrow R^{\beta}_{a\gamma}$

4. Take the product of all the above mentioned $t_{\ldots}$, $t^{\cdot\cdot}$, and $R^{\cdot}_{\cdot\cdot}$ symbols.

5. Sum over $\alpha, \beta, \ldots$, and a, b, $\ldots$, and call the result $C_{\mathcal{G}}(D)$.

For example, if D is the diagram in figure 5, then (summation understood)

$$C_{\mathcal{G}}(D) = t_{a'b'c'} t^{a'a} t^{b'b} t^{c'c} R^{\beta}_{a\gamma} R^{\gamma}_{b\alpha} R^{\alpha}_{c\beta} \tag{1}$$

Well-definedness We will now check that $C_{\mathcal{G}}(D)$ is independent of the choices of bases that were made. Clearly, $C_{\mathcal{G}}(D)$ is independent of the choice of $\{r^{\alpha}\}$ — as is demonstrated in (1) the representation R appears only through matrix traces of the form

$$Tr\ R(\mathcal{G}_a)R(\mathcal{G}_b)R(\mathcal{G}_c).$$

Suppose that $\{\bar{\mathcal{G}}_{\bar{a}}\}$ is a different basis of $\mathcal{G}$. One can define $\bar{t}^{\bar{a}\bar{b}}$, $\bar{t}_{\bar{a}\bar{b}\bar{c}}$, and $\bar{R}^{\alpha}_{\bar{a}\beta}$ with respect to this new basis, and use these tensors to define $\bar{C}_{\mathcal{G}}(D)$. We will show now that $\bar{C}_{\mathcal{G}}(D) = C_{\mathcal{G}}(D)$. The two bases are related by some linear transformation — that is to say, there exists a matrix $\{M^{\bar{a}}_{a}\}$ for which

$$\bar{\mathcal{G}}_{\bar{a}} = M^{a}_{\bar{a}} \mathcal{G}_a$$

One can check rather easily that the new tensors are given by the old ones through the following formulae:

$$\begin{aligned} \bar{t}_{\bar{a}\bar{b}} &= M^{a}_{\bar{a}} M^{b}_{\bar{b}} t_{ab} \\ \bar{t}^{\bar{a}\bar{b}} &= (M^{-1})^{\bar{a}}_{a} (M^{-1})^{\bar{b}}_{b} t^{ab} \\ \bar{t}_{\bar{a}\bar{b}\bar{c}} &= M^{a}_{\bar{a}} M^{b}_{\bar{b}} M^{c}_{\bar{c}} t_{abc} \\ \bar{R}^{\alpha}_{\bar{a}\beta} &= M^{a}_{\bar{a}} R^{\alpha}_{a\beta} \end{aligned}$$

where $(M^{-1})^{\bar{a}}_{a}$ is the inverse matrix of $M^{a}_{\bar{a}}$. It is now easy to see that when these expressions for $\bar{t}^{\bar{a}\bar{b}}$, $\bar{t}_{\bar{a}\bar{b}\bar{c}}$, and $\bar{R}^{\alpha}_{\bar{a}\beta}$ are combined together to form $\bar{C}_{\mathcal{G}}(D)$, every matrix $(M^{-1})^{\bar{a}}_{a}$ cancels every $M^{a}_{\bar{a}}$.

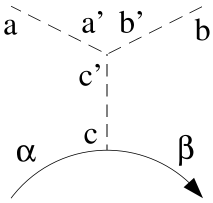
3 Relations between the $C_{\mathcal{G}}(D)$'s

So far, we used the fact that the tensors that went into the construction of $C_{\mathcal{G}}(D)$ came from a Lie algebra and satisfied certain relations only in a very mild way — in checking that $t_{abc} = t_{bca} = t_{cab}$. We will now see what relations among the $C_{\mathcal{G}}(D)$'s can be deduced from the relations that t^{ab}, t_{abc}, and $R^{\alpha}_{a\beta}$ are known to satisfy.

First, a slight generalization. Using more or less the same procedure as before we can assign to every non-closed diagram D, which is allowed to have propagators with "free" ends and non-closed Wilson lines, a tensor

$$\mathcal{T} = \mathcal{T}(D) \in \mathcal{G}^{\otimes n} \otimes \bigotimes_{i=1}^{J} \left(R_i \otimes \bar{R}_i \right). \tag{2}$$

Here n is the number of propagators with *free* ends, $R_1, \ldots R_J$ are the representations corresponding to the non-closed Wilson lines, and the $\bar{R}_i$'s are their duals. It is clear how to define $\mathcal{T}$ — one just needs to follow the same steps as in the definition of $C_{\mathcal{G}}$, and as D is not closed some of the indices will appear only once in the resulting expression and instead of being summed over these indices will serve as the indices of the tensor $\mathcal{T}$. For example:



$$\longrightarrow \qquad \mathcal{T}^{\alpha,ab}_{\beta} = \sum_{a',b',c',c} t^{aa'} t^{bb'} t^{cc'} t_{b'a'c'} R^{\alpha}_{c\beta} \in \mathcal{G}^{\otimes 2} \otimes R \otimes \bar{R}$$

Claim 1 *The two diagrams in figure 2 correspond to tensors which are the negatives of each other.*

Proof The is simply the fact that the Lie bracket is anti-symmetric.

□

Claim 2 *Let the diagrams S, T, and U be as in figure 4. Then the tensors corresponding to them satisfy:*

$$\mathcal{T}(S) = \mathcal{T}(T) - \mathcal{T}(U) \tag{3}$$

Proof This is simply the fact that R is a representation. That is, that $R([\mathcal{G}_a, \mathcal{G}_b]) = R(\mathcal{G}_a)R(\mathcal{G}_b) - R(\mathcal{G}_b)R(\mathcal{G}_a)$.

□

Claim 3 *Let the diagrams I, H, and X be as in figure 3. Then the tensors corresponding to them satisfy:*

$$\mathcal{T}(I) = \mathcal{T}(H) - \mathcal{T}(X) \tag{4}$$

Proof Translating I, H, and X into their corresponding tensors, it is easy to see that this is simply the Jacobi identity! (In fact, this claim can be regarded as a particular case of the previous one, asserting that the adjoint action of a Lie-algebra on itself is a representation).

□

Sewing. Given two open diagrams A and B and a (partial) correspondence φ between their open ended lines which sends a propagator to a propagator and an ingoing (outgoing) Wilson line to an outgoing (ingoing) Wilson line labeled by the same representation, one can define their *join* $A\#B$ to be the diagram obtained by sewing the external lines of A with those of B according to the correspondence φ. It is also possible to *sew* $\mathcal{T}(A)$ to $\mathcal{T}(B)$ by contracting their indices as dictated by φ, (using t_{ab} to lower the propagator indices). It is clear that the resulting $\mathcal{T}(A)\#\mathcal{T}(B)$ will equal $\mathcal{T}(A\#B)$. In particular, if $A\#B$ is a closed diagram, then $C_{\mathcal{G}}(A\#B) = \mathcal{T}(A)\#\mathcal{T}(B)$. (See figure 6).

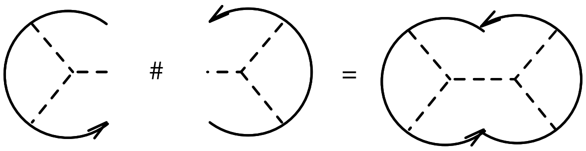


Figure 6. Sewing two diagrams.

Thus (4) and (3) can be used to derive relations between closed diagrams — (4) says that if three diagrams $\bar{I}$, $\bar{H}$ and $\bar{X}$ are identical outside of a small domain in which they look like the diagrams I, H, and X of figure 3, then they satisfy

$$C_{\mathcal{G}}(\bar{I}) = C_{\mathcal{G}}(\bar{H}) - C_{\mathcal{G}}(\bar{X}). \tag{5}$$

Similarly, (3) implies

$$C_{\mathcal{G}}(\bar{S}) = C_{\mathcal{G}}(\bar{T}) - C_{\mathcal{G}}(\bar{U}). \tag{6}$$

The last two relations show that $D \to C_{\mathcal{G}}(D)$ is a weight system in the sense of section 1.

Lemma 3.1 *For any open diagram D, $\mathcal{T} = \mathcal{T}(D)$ is an invariant tensor (with respect to the natural action of $\mathcal{G}$ on each of the components in (2)).*

Proof The reason why this lemma is true, is that $\mathcal{T}$ can be seen as the contraction of a collection of invariant tensors — the $t_{...}$, the $t^{\cdot\cdot}$ and the $R^{\cdot}_{\cdot\cdot}$ are all invariant. This statement can be translated into a combinatorial invariance proof. I will just sketch this proof here, and supplement this sketch with a simple example — figure 7.

For simplicity, I will disregard $\pm$ signs here. Say D has n internal vertices. Pick a point P outside of D and consider the $3n$ diagrams obtained by connecting P using a propagator to each of the three lines emanating from each of the n vertices in D. Let $\mathcal{D}$ be the sum of the tensors corresponding to these $3n$ diagrams. Each internal line in D has two terms corresponding to it in $\mathcal{D}$ coming from the two vertices at the ends of that line, and with the proper choice of signs these two terms exactly cancel. The only diagrams that still contribute to $\mathcal{D}$ are those in which P is connected to an external line, and, if P is marked by a, *these are exactly the diagrams that represent the variation of D with respect to $\mathcal{G}_a$*.

On the other hand, the relations (4) and (3) show that each group of three diagrams made by connecting P to the three lines emanating from a single propagator corresponds to tensors that add up to 0. $\mathcal{D}$ is just a sum of such groups, and this concludes the proof. (See figure 7).

□

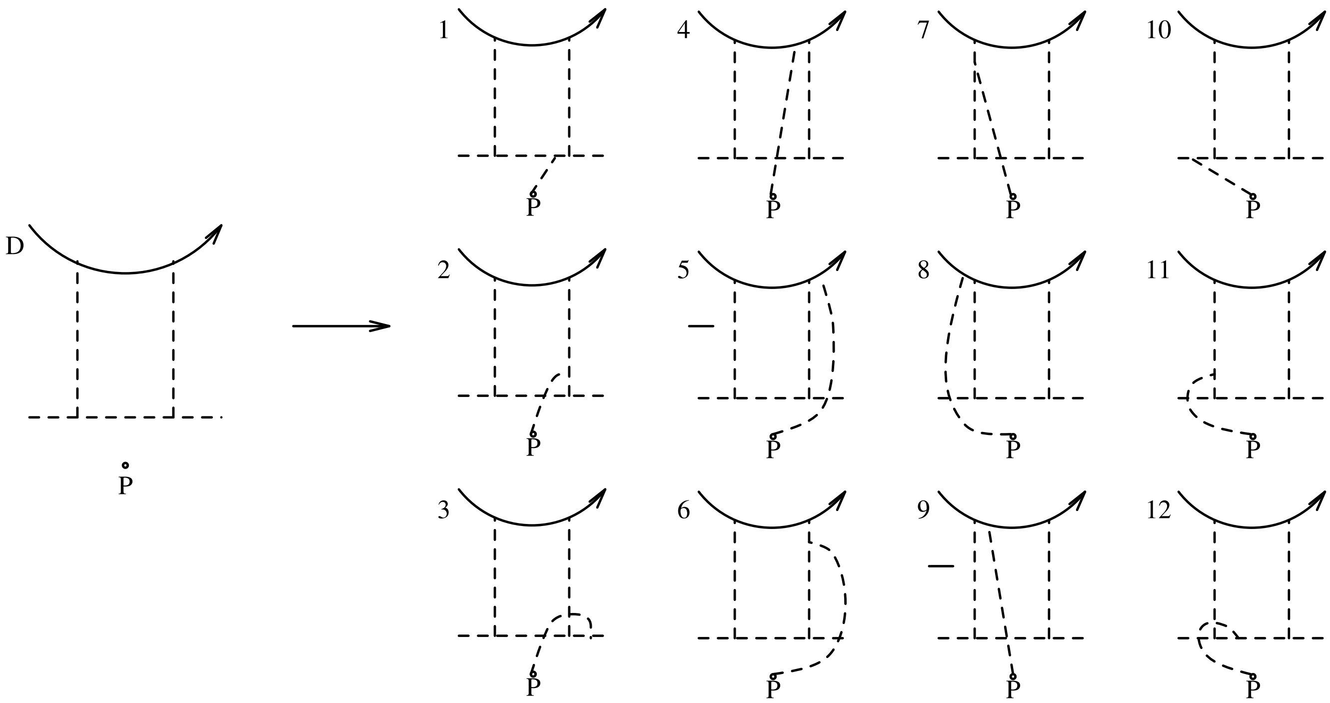
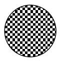
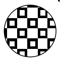


Figure 7. A simple invariance proof — the tensor $\mathcal{D}$ is the sum of **1-12**. Relation IHX shows that $\mathbf{1}+\mathbf{2}+\mathbf{3}=\mathbf{10}+\mathbf{11}+\mathbf{12}=0$, relation STU shows that $\mathbf{4}+\mathbf{5}+\mathbf{6}=\mathbf{7}+\mathbf{8}+\mathbf{9}=0$, claim 1 shows that $\mathbf{1}+\mathbf{12}=\mathbf{2}+\mathbf{6}=\mathbf{7}+\mathbf{11}=0$, and $\mathbf{4}+\mathbf{9}=0$ by the choice of signs. It follows that $\mathbf{3}+\mathbf{5}+\mathbf{8}+\mathbf{10}=0$. This is exactly the fact that $\mathcal{T}$ is an invariant tensor.

Remark The behavior of $D \to \mathcal{T}(D)$ under sewing means that we've actually defined a topological Quantum Field Theory of dimension 1, satisfying Segal's axioms (see [1, 12]). Lemma 3.1 shows that the vector space assigned by our QFT to $n + 2J$ points, n of which labeled '$\mathcal{G}$', J labeled $R_1, \ldots R_J$, and J labeled $\bar{R}_1, \ldots \bar{R}_J$, is the space of invariant tensors in

$$\mathcal{G}^{\otimes n} \otimes \bigotimes_{i=1}^{J} \left(R_i \otimes \bar{R}_i \right).$$

Every diagram D with $n + 2J$ free ends (of the appropriate kinds) gives a vector $\mathcal{T}(D)$ in that vector space.

Lemma 3.2 *If the representation R is irreducible, the factorization property illustrated in figure 8 holds. (In that figure, the blobs*  *and*  *simply represent arbitrary subdiagrams with an arbitrary number of connections to the Wilson loop).*

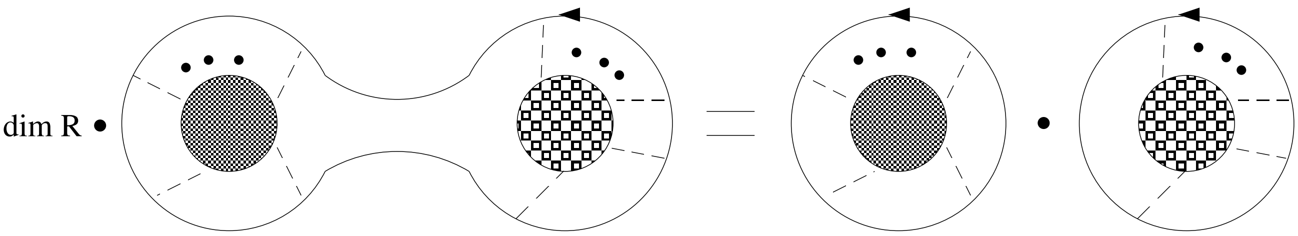


Figure 8. The factorization property.

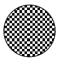
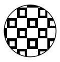
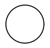
Proof Clearly, the two sides of the equation in figure 8 represent two ways of contracting the tensors $A^{\alpha}{}_{\beta}$ and $B^{\beta}{}_{\alpha}$ corresponding to the two open diagrams obtained by removing the "bridge" in the left hand side of that equation. But from lemma 3.1 and the irreducibility of R it follows that A and B must be multiples of the identity matrix:

$$A^{\alpha}{}_{\beta} = a\delta^{\alpha}{}_{\beta} \quad ; \quad B^{\beta}{}_{\alpha} = b\delta^{\beta}{}_{\alpha}.$$

This reduces figure 8 to the trivial assertion

$$da\delta^{\alpha}{}_{\beta} b\delta^{\beta}{}_{\alpha} = a\delta^{\alpha}{}_{\alpha} b\delta^{\beta}{}_{\beta}.$$

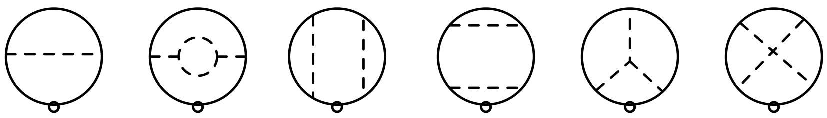
$\square$

Remark taking the blobs  and  to be empty implies that $C_{\mathcal{G}}($  $) = \dim R = d$.

4 Taking the logarithm

In this section we will assume that $\mathbf{F}$ is a field of characteristic zero and that R is an *irreducible* representation of $\mathcal{G}$.

Definition 1 *Let $\mathcal{A}$ be the vector space of (infinite) formal linear combinations (with coefficients in $\mathbf{F}$) of (graph-) isomorphism types of closed diagrams having $I = 1$, (i.e. containing exactly one Wilson loop), with a pre-chosen base point on that loop. For convenience, we will exclude the trivial diagram*  *from $\mathcal{A}$. For example, here are the six simplest generators of $\mathcal{A}$:*



In fact, $\mathcal{A}$ can be made into an algebra; the product of $A \in \mathcal{A}$ and $B \in \mathcal{A}$ will essentially be the sum of all the possible ways of merging them into a single diagram:

Definition 2 *Let A be a generator of $\mathcal{A}$, and let $a_1, a_2, \ldots, a_n$ be the list of $R^2\mathcal{G}$ vertices in A, in the order they are encountered when one travels along the loop consistently with its orientation and beginning from the base point. Let B be another generator of $\mathcal{A}$, and define $b_1, b_2, \ldots, b_k$ in the same way. Let $\mathcal{P}$ be the set of all possible linear orderings of n "a" symbols and m "b" symbols. For every $P \in \mathcal{P}$ define $[AB]_P$ to be the diagram obtained by marking a based Wilson loop with a's and b's following their order in P, and connecting diagrams A and B (minus their respective loops) to that Wilson loop following the marks in the obvious way. Finally, define*

$$A \cdot B = \sum_{P \in \mathcal{P}} [AB]_P \, .$$

For an example, see figure 9.

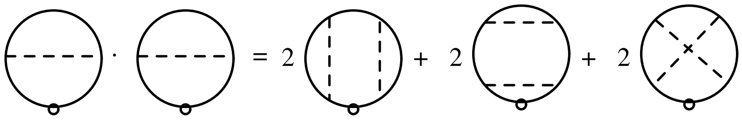


Figure 9. Taking the product in $\mathcal{A}$

Claim 4 *The algebra $\mathcal{A}$ is associative and commutative.*

□

Now let $\mathcal{Z} \in \mathcal{A}$ be

$$\mathcal{Z} = d + \sum_{\text{generators of } \mathcal{A}} C_{\mathcal{G}}(D) \cdot D, \tag{7}$$

and let $\mathcal{W} \in \mathcal{A}$ be the formal logarithm of $\mathcal{Z}$,

$$\mathcal{W} = \log \mathcal{Z},$$

given by the formal power series expansion

$$\mathcal{W} \stackrel{\text{def}}{=} \log d + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}\left(\sum_D C_{\mathcal{G}}(D)\cdot D\right)^m}{md^m}. \tag{8}$$

Notice that the order of $A\cdot B$ is always bigger than that of A or B, and so every diagram D appears in the above infinite sum only finitely many times, and hence $\mathcal{W}$ is well defined.

Definition 3 *Define $C'_{\mathcal{G}}(D)$ to be the coefficient of D in $\mathcal{W}$. Namely, define it by the equation*

$$\mathcal{W} = \log d + \sum_D C'_{\mathcal{G}}(D)\cdot D.$$

Remark It is easy to check that the weight of a diagram is independent of the position of its base point, which was introduced only for the sake of simplifying definition 2. Therefore, base points will be suppressed from now on.

Definition 4 *Let D be a generator of $\mathcal{A}$. A 'cyclic partition' of D will be a cyclicly ordered (that is, ordered up to a rotation) partition $\mathfrak{D} = \{D_1, D_2, \ldots, D_{k(\mathfrak{D})}\}$ of the set of all propagators of D into disjoint subsets, such that for any propagator $p \in D_i$, all the propagators connected to p by a $\mathcal{G}^3$ vertex will also be in D_i. Given such a partition, we will denote by the same letter D_i the generator of $\mathcal{A}$ obtained by reinserting the Wilson loop of D around D_i.*

Claim 5 *The weight $C'_{\mathcal{G}}(D)$ of a generator D of $\mathcal{A}$ is given in the following formula:*

$$C'_{\mathcal{G}}(D) = \sum_{\text{cyclic partitions } \mathfrak{D}} \frac{(-1)^{k(\mathfrak{D})+1}}{d^{k(\mathfrak{D})}} \prod_{i=1}^{k(\mathfrak{D})} C_{\mathcal{G}}(D_i). \tag{9}$$

Proof This is simply a sum over all the possible ways of writing D as a product in $\mathcal{A}$, with the coefficients taken correctly as in (8). The fact that we are restricting our attention to "cyclic partitions" corresponds to the factor $\frac{1}{m}$ in that equation.

□

Lemma 4.1 *Let D be a generator of $\mathcal{A}$ which can be decomposed (in the sense of definition 4) into two parts such that:*

1. *The two parts can be separated from each other by cutting the Wilson loop of D at just two points.*

2. *At least one of the parts cannot be decomposed any further.*

In this case,

$$C'_{\mathcal{G}}(D) = 0. \tag{10}$$

(For an example, see figure 10).

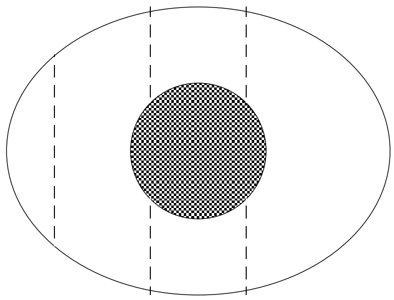


Figure 10. An example for a diagram with $C'_{\mathcal{G}}(D) = 0$

Proof Let $D = A \cup B$ be a diagram decomposed into two non-empty separated parts such that A cannot be be decomposed any further. Write

$$C'_{\mathcal{G}}(D) = \sum_{\text{cyclic partitions } \mathfrak{D}} c'(\mathfrak{D}) \quad ; \quad c'(\mathfrak{D}) = \frac{(-1)^{k(\mathfrak{D})+1}}{d^{k(\mathfrak{D})}} \prod_{i=1}^{k(\mathfrak{D})} C_{\mathcal{G}}(D_i).$$

We will prove (10) by finding a fixed point free involution $\mathfrak{D} \to \rho\mathfrak{D}$ of the set of all cyclic partitions of D for which $c'(\rho\mathfrak{D})$ is always the negative of $c'(\mathfrak{D})$.

Let $\mathfrak{D} = \{D_1, D_2, \ldots, D_{k(\mathfrak{D})}\}$ be a cyclic partition of D. There are two possibilities:

1. *A is one of the D_i's.* In this case, define $\rho\mathfrak{D}$ to be the cyclic partition obtained by adjoining A to the component of D preceding it in $\mathfrak{D}$. It is clear that $k(\rho\mathfrak{D}) = k(\mathfrak{D}) - 1$, and therefore using lemma 3.2 we find $c'(\rho\mathfrak{D}) = -c'(\mathfrak{D})$.

2. *A is properly contained in one of the D_i's.* We may assume that A is properly contained in D_1. Define $\rho\mathfrak{D} = \{D_1 - A, A, D_2, \ldots, D_{k(\mathfrak{D})}\}$. It is clear that $k(\rho\mathfrak{D}) = k(\mathfrak{D}) + 1$, and therefore using lemma 3.2 we find $c'(\rho\mathfrak{D}) = -c'(\mathfrak{D})$.

It is clear that ρ is a fixed point free involution.

□

Remark It is easy to show that the second requirement of the above lemma is superfluous — even if one of the parts of D is still decomposable one can always use relation STU to express that part as a sum of open diagrams, each of which is either 'less decomposable' or 'more separable' (i.e. can be separated in the sense of the first requirement of the above lemma into two smaller parts).

Claim 6 *The relations (5) and (6) hold for the $C'_{\mathcal{G}}(D)$'s as well:*

$$C'_{\mathcal{G}}(\bar{I}) = C'_{\mathcal{G}}(\bar{H}) - C'_{\mathcal{G}}(\bar{X}), \tag{11}$$

$$C'_{\mathcal{G}}(\bar{S}) = C'_{\mathcal{G}}(\bar{T}) - C'_{\mathcal{G}}(\bar{U}). \tag{12}$$

Proof (5) is a linear relation, and it is respected by each term in the sum (9). Therefore (11) holds. The same is true for (12), only that $\bar{T}$ and $\bar{U}$ have cyclic partitions which do not correspond to cyclic partitions of $\bar{S}$ — these are the ones in which the two propagators in T or in U of figure 4 appear in different components. There is natural correspondence ρ between those exceptional partitions of $\bar{T}$ and those of $\bar{U}$, and clearly $c'(\rho\mathfrak{D}) = c'(\mathfrak{D})$ for every exceptional partition $\mathfrak{D}$ of $\bar{T}$. The minus sign in (12) then shows that these exceptional partitions can be disregarded.

$\square$

Remark The algebra structure of $\mathcal{A}$ can be used to define an algebra structure on the space $\mathcal{C}$ of all weight systems. Let the *generating function* $\mathcal{Z}_C$ of a weight system C be as in (7),

$$\mathcal{Z}_C = d + \sum_{\text{generators of } \mathcal{A}} C(D) \cdot D,$$

and for $C_{1,2} \in \mathcal{C}$ define their product $C_1 \cdot C_2$ by

$$\mathcal{Z}_{C_1 \cdot C_2} = \mathcal{Z}_{C_1} \cdot \mathcal{Z}_{C_2}.$$

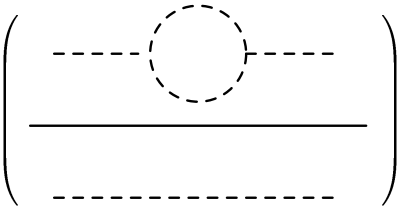
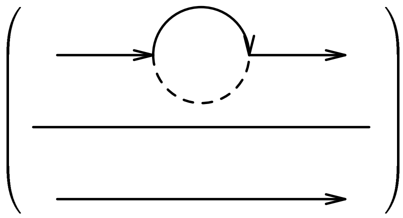
The above proof is essentially a verification of the fact that $\mathcal{Z}_{C_1 \cdot C_2}$ is indeed the generating function of a weight system that satisfies the relations IHX and STU.

5 Evaluation of some diagrams for simple algebras

In this section $\mathcal{G}$ will be a simple Lie algebra over the real or complex field, and R will be an irreducible representation of $\mathcal{G}$. In this context, it is possible to evaluate some diagrams in a relatively simple way.

The key point is that under the above conditions, the spaces of invariant tensors in $\mathcal{G} \otimes \mathcal{G}$ and in $R \otimes \bar{R}$ are both one-dimensional, and therefore one can speak of 'ratios' of invariant tensors in $\mathcal{G} \otimes \mathcal{G}$ or in $R \otimes \bar{R}$.

Definition 5 *The constants r and g are given by the following ratios:*

$g =$  $r =$  . (13)

(Notice that by lemma 3.1 the above tensors are all invariant).

In the following few lines, we see how the relations from the previous section can be used to evaluate $C_{\mathcal{G}}$ for all closed diagrams with a single Wilson loop and orders smaller than three. For brevity, we omit the symbol $C_{\mathcal{G}}$ below.

 $= d$ by the remark after lemma 3.2 (14)

 $= r$  $= dr$ by (13) and (14) (15)

 $= g$  $= dgr$ by (13) and (15) (16)

 $= r$  $= dr^2$ (17)

 $= \frac{1}{2}\Big($  $-$  $\Big) = \frac{1}{2}$  $= \frac{1}{2}dgr$ by (3) (18)

 $=$  $-$  $= dr\left(r - \frac{1}{2}g\right)$ (19)

Similarly:

 $= dgr^2$

 $= dr^3$

 $= dr^3$

 $= dr^2(r - \frac{1}{2}g)$

 $= \frac{1}{2}dgr^2$

 $= \frac{1}{2}dg^2r$

 $= \frac{1}{4}dg^2r$

 $= \frac{1}{2}dgr(r - \frac{1}{2}g)$

 $= dr(r - \frac{1}{2}g)^2$

 $= dgr(r - \frac{1}{2}g)$

 $= \frac{1}{4}dg^2r$

 $= 0$

 $= dr(r - \frac{1}{2}g)(r - g)$

 $= dg^2r$

It is now possible to use these results together with (9) and get:

$C'_{\mathcal{G}}($  $) = r$

$C'_{\mathcal{G}}($  $) = gr$

$C'_{\mathcal{G}}($  $) = \frac{1}{2}gr$

$C'_{\mathcal{G}}($  $) = -\frac{1}{2}gr$

$C'_{\mathcal{G}}($  $) = \frac{1}{2}g^2r$

$C'_{\mathcal{G}}($  $) = \frac{1}{4}g^2r$

$C'_{\mathcal{G}}($  $) = -\frac{1}{4}g^2r$

$C'_{\mathcal{G}}($  $) = \frac{1}{4}g^2r$

$C'_{\mathcal{G}}($  $) = -\frac{1}{2}g^2r$

$C'_{\mathcal{G}}($  $) = \frac{1}{4}g^2r$

$C'_{\mathcal{G}}($  $) = \frac{1}{2}g^2r$

$C'_{\mathcal{G}}($  $) = g^2r$

It is easy to check that all the other diagrams of order ≤ 3 have a vanishing $C'_{\mathcal{G}}$.

Unfortunately, there are some order four (and higher) diagrams that cannot be evaluated using these techniques. One such diagram is  .

The following table contains the values of d, g, and r for some classical Lie algebras with their defining representations (and t_{ab} taken to be the matrix trace in those representations):

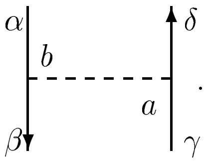
$\mathcal{G}$	R	d	g	r
$sl(N, \mathbf{C})$	$\mathbf{C}^N$	N	$2N$	$\frac{N^2-1}{N}$
$so(N, \mathbf{C})$	$\mathbf{C}^N$	N	$N-2$	$\frac{N-1}{2}$
$sp(N, \mathbf{C})$	$\mathbf{C}^{2N}$	$2N$	$2(N+1)$	$N+\frac{1}{2}$

Remark One can check that if $\mathcal{G}$ is a real Lie algebra and $\mathcal{G}_{\mathbf{C}}$ is its complexification then $C_{\mathcal{G}} \equiv C_{\mathcal{G}_{\mathbf{C}}}$. Therefore the above table can be used to evaluate d, g, and r for any of the real forms of $sl(N, \mathbf{C})$, $so(N, \mathbf{C})$, or $sp(N, \mathbf{C})$ in their defining representations.

6 Complete evaluation for the classical algebras

By the remark at the end of the previous section, to calculate $C_{\mathcal{G}}$ for the classical algebras (in their defining representations) it is enough to consider the four complex classical algebras.

The first step is to use relation STU repeatedly, with each usage reducing the number of $\mathcal{G}^3$ vertices by one, until we are left with a diagram D that has no $\mathcal{G}^3$ vertices. The basic building block of such diagrams is the tensor

$\mathcal{T}^{\alpha\gamma}_{\beta\delta} =$  .

This tensor will be evaluated explicitly for each of the complex classical algebras, and the results will turn out to have representations in terms of diagrams that have no propagators in them. Using this repeatedly, we are left with disjoint unions of circles which again are easy to evaluate explicitly.

I will show in detail the computations for $so(N, \mathbf{C})$, and just state the results for $gl(N, \mathbf{C})$, $sl(N, \mathbf{C})$, and $sp(N, \mathbf{C})$.

The algebra $so(N, \mathbf{C})$. A convenient choice of generators for $so(N, \mathbf{C})$ are the $N \times N$ matrices M_{ij} $(i < j)$, given by

$$(M_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}.$$

That is, the ij entry of M_{ij} is $+1$, the ji entry of M_{ij} is -1, and all other entries of M_{ij} are zero. The invariant bilinear form that we pick on $so(N, \mathbf{C})$ is the matrix trace in the defining representation, and so

$$t_{(ij)(kl)} \stackrel{\text{def}}{=} tr(M_{ij}M_{kl}) = -2\delta_{ik}\delta_{jl}.$$

Inverting the $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$ matrix $t_{(ij)(kl)}$ we get

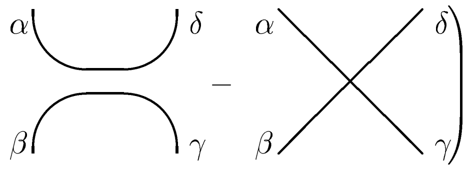
$$t^{(ij)(kl)} = -\frac{1}{2}\delta^{ik}\delta^{jl}, \tag{20}$$

and so

$$\mathcal{T}^{\alpha\gamma}_{\beta\delta} = \sum_{i<j;k<l} t^{(ij)(kl)}(M_{ij})_{\alpha\beta}(M_{kl})_{\gamma\delta}. \tag{21}$$

Using (20) and some algebraic manipulations we can simplify (21), and then represent it by a diagram:

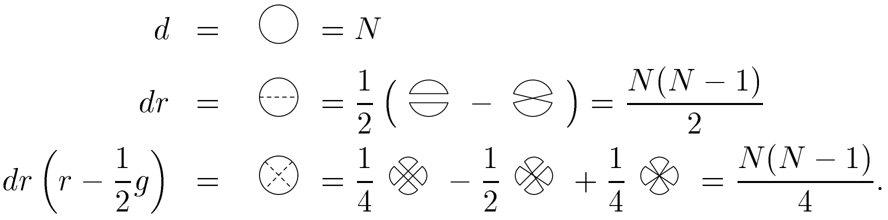
$$(21) = \frac{1}{2}(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}) = \frac{1}{2}\left(\quad - \quad \right). \tag{22}$$



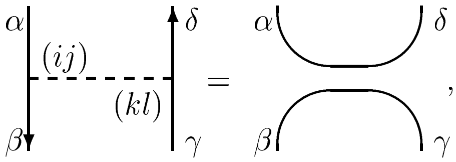
The last thing to note is that

$$C_{so(N,\mathbf{C})}(k \text{ disjoint circles}) = N^k.$$

Example For $so(N,\mathbf{C})$ in its defining representation we can calculate d, r, and g using: (suppressing the '$C_{so(N,\mathbf{C})}$' symbols)



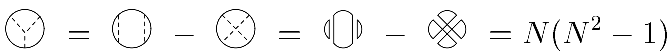
The algebra $gl(N,\mathbf{C})$. Similar considerations lead to the even simpler rule



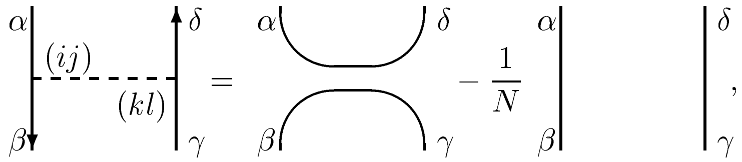
while retaining

$$C_{gl(N,\mathbf{C})}(k \text{ disjoint circles}) = N^k.$$

Example For $gl(N,\mathbf{C})$ in its defining representation



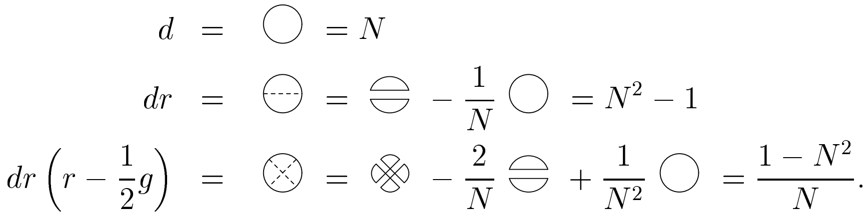
The algebra $sl(N,\mathbf{C})$. The rule here is the so-called "Fierz identity",



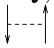
with the usual

$$C_{sl(N,\mathbf{C})}(k \text{ disjoint circles}) = N^k.$$

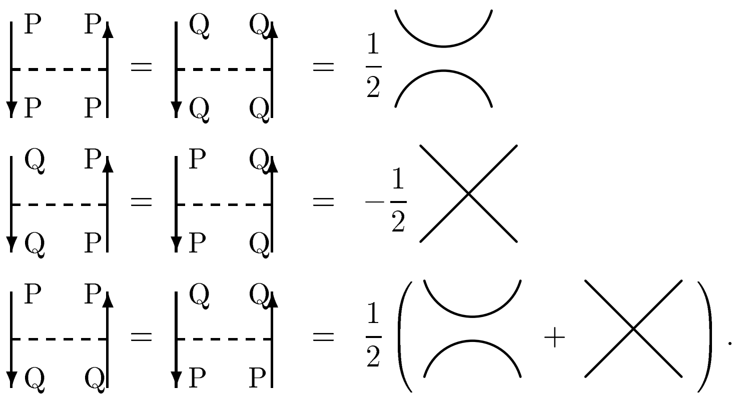
Example For $sl(N,\mathbf{C})$ in its defining representation we can calculate d, r, and g using:



The algebra $sp(N,\mathbf{C})$. This is the most complicated case. Let D be a diagram with no $\mathcal{G}^3$ vertices. The computation of $C_{sp(N,\mathbf{C})}(D)$ now proceeds in two steps:

1. Mark each Wilson loop segment in D with either the symbol P or the symbol Q, in such a way that the number of P's entering each subdiagram of D of the form  is equal to the number of P's leaving it. (Remember that the Wilson loops are *directed*).

2. Simplify D using the following rules:

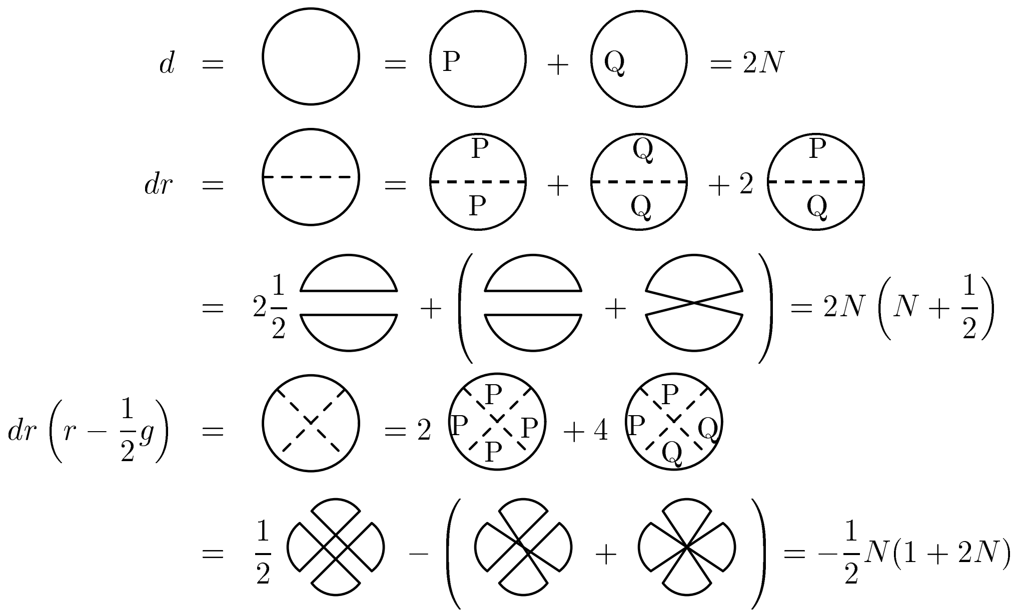


3. Similarly to the usual,

$$C_{sp(N,\mathbf{C})}(k \text{ disjoint } \textit{marked} \text{ circles}) = N^k.$$

(Notice that this time $\dim R = 2N \neq N$).

Example For $sp(N, \mathbf{C})$ in its defining representation we can calculate d, r, and g using:



Exercise The reader might find it amusing to verify that $C_{sp(1,\mathbf{C})} \equiv C_{sl(2,\mathbf{C})}$, as expected from the isomorphism $sp(1, \mathbf{C}) \cong sl(2, \mathbf{C})$. Notice that $C_{so(3,\mathbf{C})}$ is not equal to $C_{sp(1,\mathbf{C})}$ (or $C_{sl(2,\mathbf{C})}$) because their defining representations are not the same.

7 The Vassiliev knot invariants

In [9] Vassiliev considered the space $\mathcal{M}$ of all the possible embeddings of the oriented circle S^1 in an oriented $\mathbf{R}^3$ as a subspace of the space of all smooth maps $S^1 \to \mathbf{R}^3$, analyzed the possible singularities of such maps, and using that information constructed a filtration of $\mathcal{M}$ and a spectral sequence that converges to its cohomology. The connected components of $\mathcal{M}$ correspond simply to oriented knot types, and therefore each element of $H^0(\mathcal{M})$ is a knot invariant. Vassiliev then uses his topological machinery to partially compute $H^0(\mathcal{M})$, and based on his machinery, Birman and Lin [4] arrived at the following properties which a numerical invariant V_i of oriented knots that comes from the i's level of Vassiliev's filtration has to satisfy:

1. V_i has an extension (which I will also denote by V_i) to an invariant of smooth immersed circles, which are allowed to have finitely many transversal self-intersection. We will call such immersed circles *embedded graphs.*

2. $V_i(\bigcirc) = 0$.

3. Overcrossings, undercrossings and self-intersections are related by:
$$V_i(\diagup\!\!\!\!\diagdown) - V_i(\diagdown\!\!\!\!\diagup) = V_i(\times). \tag{23}$$
This relation will be called the *flip* relation. (As usual in knot theory, when we write $\diagup\!\!\!\!\diagdown$, $\diagdown\!\!\!\!\diagup$ or $\times$, we think of them as parts of bigger graphs which are identical outside of a small sphere, inside of which they look as in the figures).

4. If a graph G has *more* than i self-intersections, then $V_i(G) = 0$.

The third and fourth properties taken together imply that if a graph G has exactly i self-intersection, than $V_i(G)$ depends only on the *abstract* graph underlying G, and not on its embedding. Such a graph will be called *saturated.* A simple way of representing such a graph is by the diagram underlying it, which is obtained by drawing a circle in the plane corresponding to the parameterization of G, and connecting using a dashed line every two points of that circle which are identified in G. For an example, see figure 11.

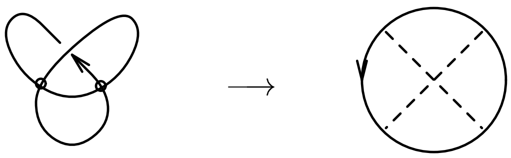


Figure 11. The diagram corresponding to a saturated graph with $i = 2$

Example A somewhat tautological example is easily derived from the Conway polynomial [5, 8]. Fix $i > 0$, let G an embedded graph with j self-intersections, and let $K_1, \ldots, K_{2^j}$ to be the 2^j possible *resolutions* of G — the 2^j knots obtained by replacing each of the j

self-intersections in G by either an overcrossing or an undercrossing. Let $\Gamma(K)(z)$ be the Conway polynomial of a knot K, and define

$$V_i^\Gamma(G) \stackrel{\text{def}}{=} \text{coefficient of } z^i \text{ in } \sum_{m=1}^{2^j} (-1)^{\#\text{ of new undercrossings in } K_m} \cdot \Gamma(K_m)(z). \tag{24}$$

I have already defined V_i^Γ for graphs, and there is nothing to check for property 1. Property 2 is the fact that $\Gamma\left(\bigcirc\right) = 1$ is independent of z, and property 3 is trivial from the definition (24). By the defining relation of the Conway polynomial

$$\Gamma\left(\diagup\!\!\!\!\diagdown\right) - \Gamma\left(\diagdown\!\!\!\!\diagup\right) = z \cdot \Gamma\left(\asymp\right)$$

and property 3, it follows that

$$V_i^\Gamma\left(\times\right) = V_{i-1}^\Gamma\left(\asymp\right),$$

and this proves that if $j > i$ then $V_i^\Gamma(G) = 0$, as required in property 4. Using the results of the previous section one can check that if G is a saturated graph and D is its corresponding diagram, then $V_i^\Gamma(G)$ is equal to the coefficient of N in $C_{gl(N,\mathbf{C})}(D)$.

We saw that underlying the Vassiliev invariants there is an assignment of weights to a certain collection of diagrams, $D \to V_i(D)$, just like the assignments $C_\mathcal{G}$ and $C'_\mathcal{G}$. The Vassiliev assignments are not arbitrary — they have to satisfy certain consistency conditions: (These conditions were first written explicitly by Birman and Lin in [4])

Claim 7 *Whenever four diagrams S, E, W, and N differ only as shown in figure 12, their weights satisfy*

$$V_i(S) - V_i(E) = -V_i(W) + V_i(N). \tag{25}$$

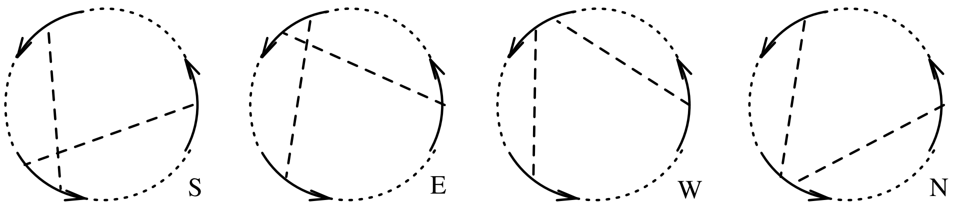


Figure 12. The diagrams S, E, W, and N. (The dotted arcs represent parts of the diagrams that are not shown in the figure. These parts are assumed to be the same in the four diagrams)

Proof Let SW be the almost saturated (i.e. having $i-1$ self-intersections) graph shown (partially) in figure 13. Pieces of the x and y axes near the origin serve as arcs in that graph, as well as a third line z' parallel to the z axis but transversing the $x-y$ plane South-West of the origin. Let NE be the same, only with the third line z' moved to transverse the $x-y$ plane North-East of the origin. There are two ways to calculate $V_i(NE)$ in terms of $V_i(SW)$ and the weights of saturated graphs using the flip relation — by moving z' from SW to NE along the two dotted paths in figure 13. The two ways must yield the same answer, and therefore the four saturated graphs corresponding to z' intersecting the x and y axes South,

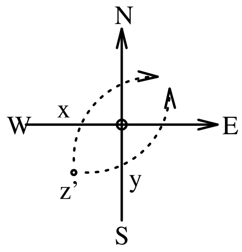
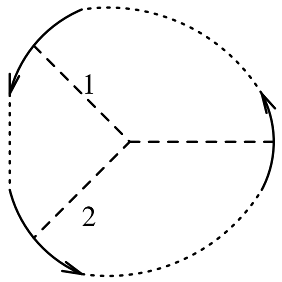


Figure 13. The graph SW and the two ways of getting from it to NE. Notice that z' is perpendicular to the plane and therefore appears as a single dot.

East, West and North of the origin have diagrams whose weights are related. With the sign convention of (23), this relation is seen to be (25).

□

It is easy to see that the weight systems $C_{\mathcal{G}}$ and $C'_{\mathcal{G}}$ satisfy the relation (25). Simply use the relations (6) and (12) in two different ways (marked 1 and 2) on the diagram:



Claim 8 *(Birman-Lin) If a diagram D contains a dashed line whose endpoints on the circle are not separated from each other by an endpoint of any other line in D, then* $V_i(D) = 0$.

Proof An embedded graph G whose corresponding diagram is D would have a kink . By the flip relation (23), $V_i(G) = V_i(G^o) - V_i(G^u)$, where G^o (G^u) is a version of G in which the kink was resolved to an overcrossing (undercrossing). But G^o and G^u are isotopic, and therefore $V_i(G) = 0$.

□

It is a trivial consequence of lemma 4.1 that The weights $C'_{\mathcal{G}}$ satisfy the relation in claim 8.

We have just solved a problem posed by Birman and Lin in [4] — to find non-trivial solutions to the relations in the last two claims.

Acknowledgements I wish to thank D. Abramovich, A-B. Berger, J.H. Conway, L. Merlach, E. Thibault, K. Walker and E. Witten for discussions and suggestions they made, and J.S. Birman and X-S. Lin for teaching me about the Vassiliev invariants.

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